Simple examples

Example1. In our first example the integrand f is a simple square function

$$f \ := \ x \mapsto x^2,$$

and the basic interval is $[0,1]$. We assume that the tagged division $D \equiv \{[u,\, v], x\}$ is d-fine with some not yet specified gauge d. We anticipate the value of the integral over $[u,\, v]$ to be $v^3/3 - u^3/3$ and have for some $z \in [u,\, v]$

$$\begin{aligned} W = \left| \sum_D \frac{1}{3} v^3 - \frac{1}{3} u^3 - x^2(v-u) \right| \\ \leq \left| \sum_D \left(z^2 - x^2\right)(v-u) \right| \\ \leq \left| \sum_D (z-x)\,(z+x)(v-u) \right| \\ \leq \sum_D d(x)(2|x| + d(x))(v-u). \quad (1) \end{aligned}$$

Let $er > 0$ (in this paper we use er rather than ε.) Now we choose d to satisfy $d(x)(2|x| + d(x)) = er$, that is

$$d \ := \ x \mapsto -\,|x| + \sqrt{x^2 + er}.$$

This combined with (1) gives

$$W \leq er \sum_D (v-u) \leq er.$$

For the display we have choosen (here and in subsequent examples)

$$er \ := \ 0.1$$

Example 2 The integrand f_2 is defined as

$$f_2 \ := \ x \mapsto 1 - x^4$$

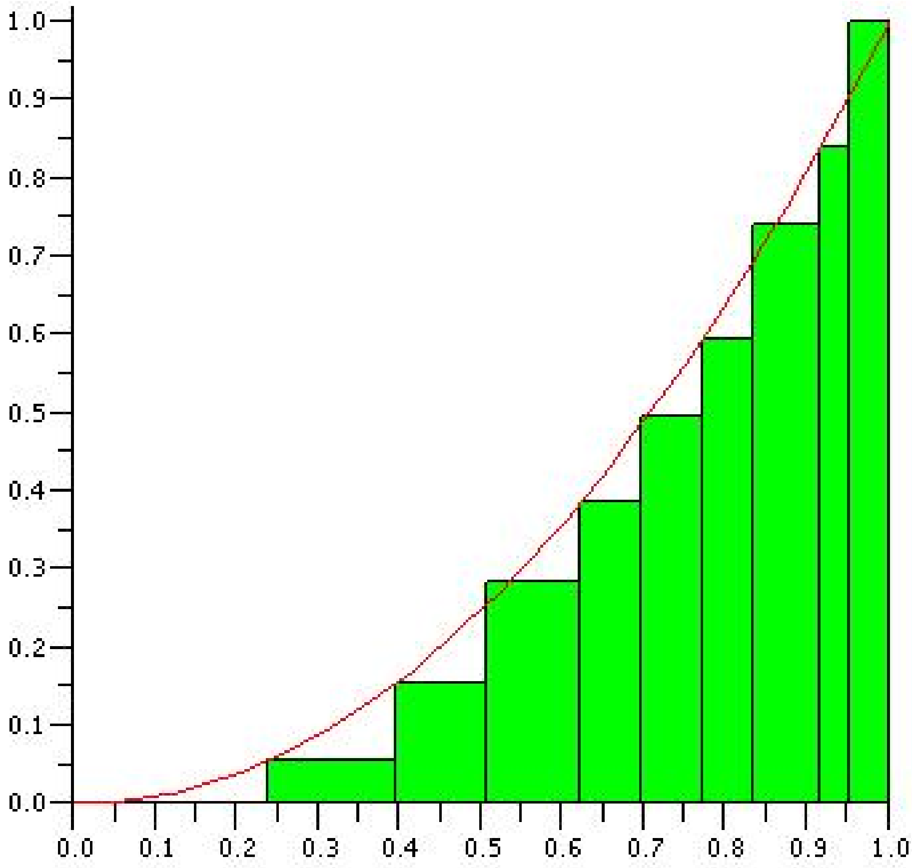

Figure 1: Example 1

We assume that the tagged division $D \equiv \{[u,\, v], x\}$ is d_2-fine with some not yet specified gauge d_2. We anticipate the value of the integral over $[u,\, v]$ to be $(1 - z^4)\,(v - u)$ for some $z \in [u,\, v]$ and have for some $w \in [u,\, v]$

$$\begin{aligned} W = \left| \sum_D \left[\left(1 - z^4\right)(v-u) - \left(1 - x^4\right)(v-u) \right] \right| \\ \le \left| \sum_D \left(z^4 - x^4\right)(v-u) \right| \\ \le \left| \sum_D 4w^3 (z - x)(v - u) \right| \\ \le \sum_D 4d_2(x)(|x| + d_2(x))^3 (v - u). \end{aligned}$$

The equation $4d_2(x)(|x|+d_2(x))^3 = er$ is hard to solve for $d_2(x)$ for a general x. However for $x = 0$ we would have $d_2(0) = \sqrt[4]{0.25er}$. Since it is natural to have d_2 decreasing on $[0, 1]$ we assume that $d_2(x) \le \sqrt[4]{0.25er}$ and this leads to

$$4d_2(x)\left(|x| + \sqrt[4]{0.25er}\right)^3 \le er.$$

Consequently we choose d_2 as

$$d_2 \;:=\; x \mapsto \min\left((0.25er)^{1/4}, \frac{0.25er}{\left(|x| + (0.25er)^{1/4}\right)^3} \right)$$

Example 3 In this example the integrand and the gauge are piecewise equal to the ones from the previous examples. More precisely

$$f_3 \;:=\; x \mapsto \begin{cases} x^2 & x < 0 \\ -1 & x = 0 \\ 1 - x^4 & \text{otherwise} \end{cases}$$

and

$$d_3 \;=\; x \mapsto \begin{cases} \min\left(|x|, -|x| + \sqrt{x^2 + er}\right) & x < 0 \\ er/2 & x = 0 \\ \min\left((0.25er)^{1/4}, \dfrac{0.25er}{\left(|x| + (0.25er)^{1/4}\right)^3} \right) & \text{otherwise} \end{cases}$$

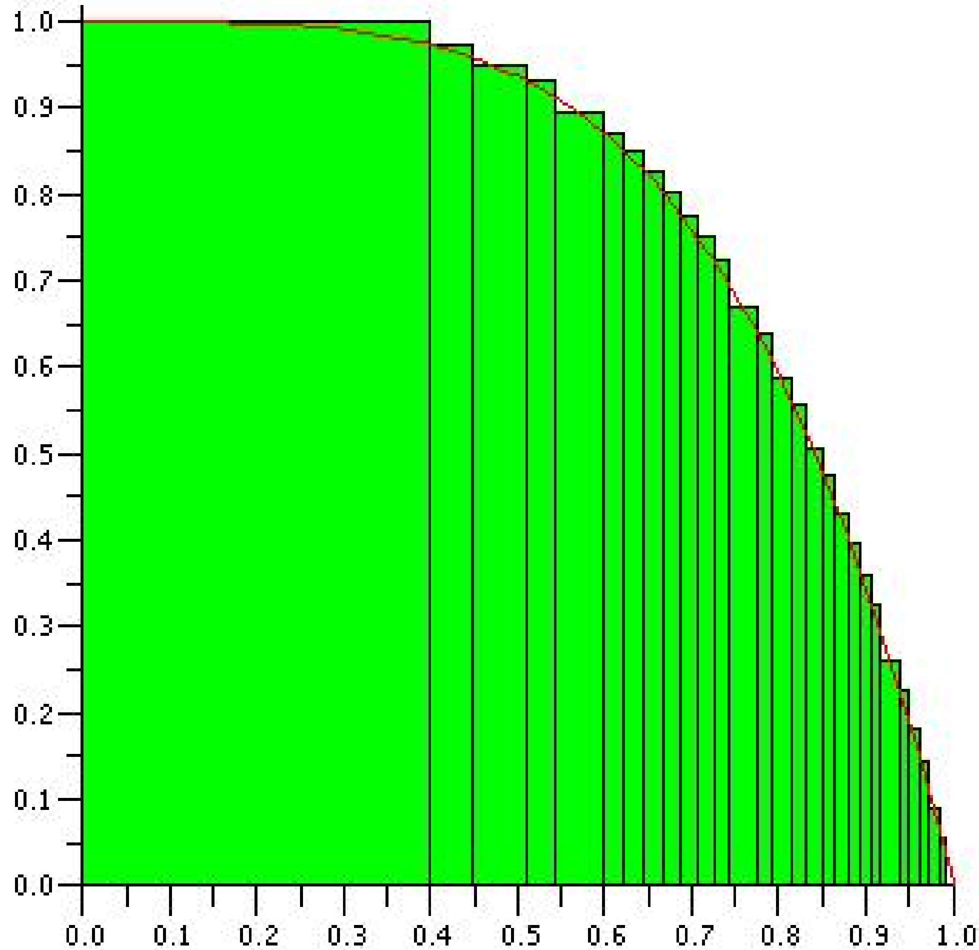

Figure 2: Example 2

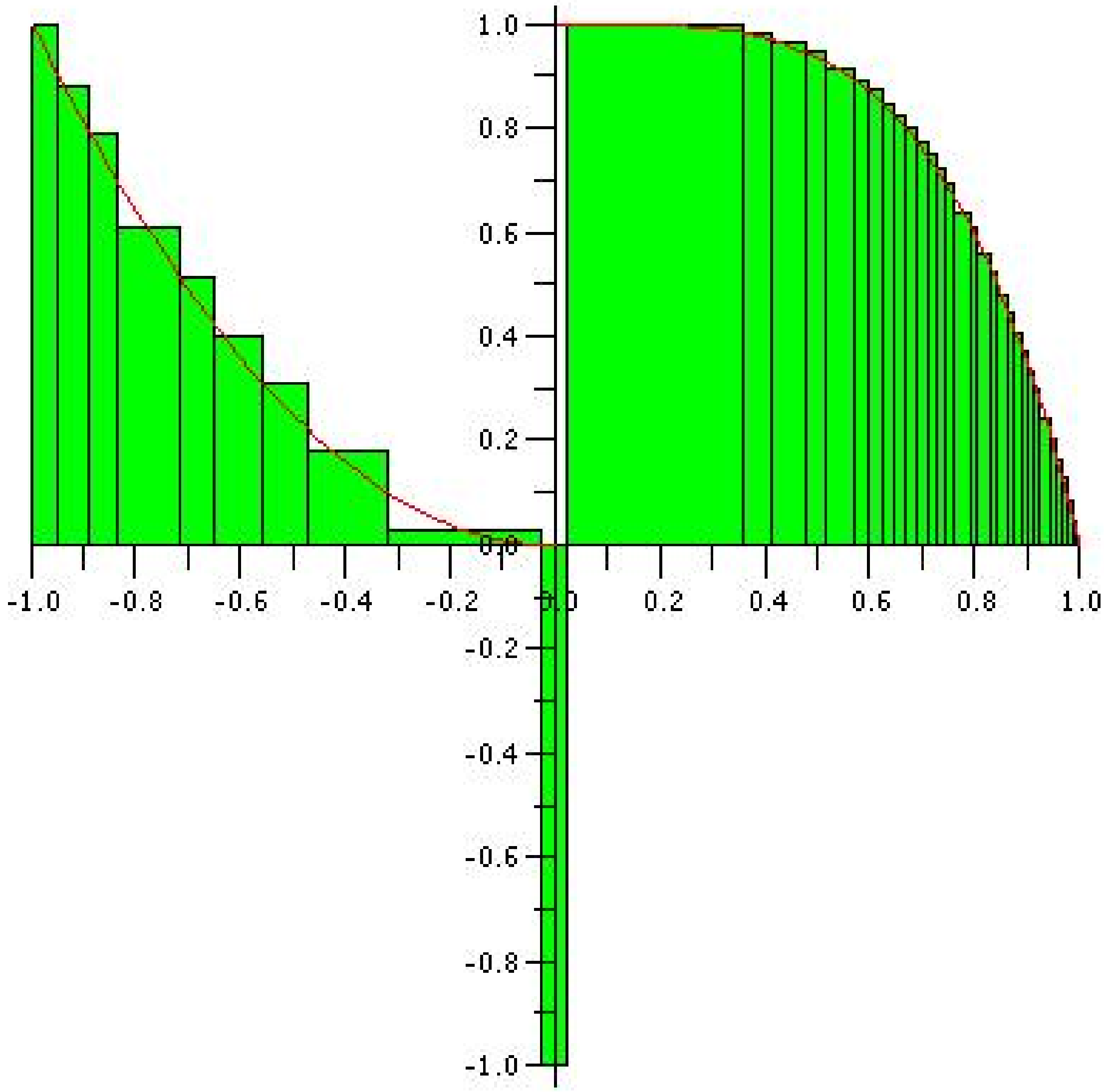

Figure 3: Example 3

Example 4 The integrand f_4 is defined as

$$f_4 := x \mapsto \left(1 + x^2\right)^{-1}$$

and the interval of integration is $[-4, 4]$. We anticipate the value of the integral over $[u, v]$ to be $\arctan(v) - \arctan(u)$. We assume that the tagged division $D \equiv \{[u, v], x\}$ is d_4-fine with some not yet specified gauge d_4. Then we have for some $z \in [u, v]$

$$W = \left|\sum_D \left(\frac{1}{1+z^2} - \frac{1}{1+x^2}\right)(v-u)\right| \\ \leq \sum_D \frac{|z-x||z+x|}{(1+z^2)(1+x^2)}(v-u). \quad (2)$$

For $|x| > 1$ we assume $d_4(x) \leq |x|/2$ and estimate

$$\frac{|z-x||z+x|}{(1+z^2)(1+x^2)} \leq \frac{(d_4(x))(2|x| + d_4(x))}{(1+x^2/4)(1+x^2)} \quad (3)$$

This leads to

$$d_4(x) = \min\left(|x|/2, -|x| + \sqrt{x^2 + er\,(1+x^2)\,(1+x^2/4)}\right) \quad (4)$$

for $|x| > 1$. On $[-1, 1]$ we simply omitt z^2 and have

$$\frac{|z-x||z+x|}{(1+z^2)(1+x^2)} \leq \frac{(d_4(x))(2x + d_4(x))}{(1+x^2)}.$$

This leads to

$$d_4(x) = -|x| + \sqrt{x^2 + er(1+x^2)} \quad (5)$$

Combining (4) and (5) gives

$$d_4 := x \mapsto \begin{cases} \min(|x|/2, -|x| + \sqrt{|x|^2 + er\,(1+x^2)\,(1+x^2/4)} & |x| > 1 \\ -|x| + \sqrt{|x| + er\,(1+x^2)} & \text{otherwise} \end{cases}$$

Finally we have

$$W \leq 3er + 2er + 3er.$$

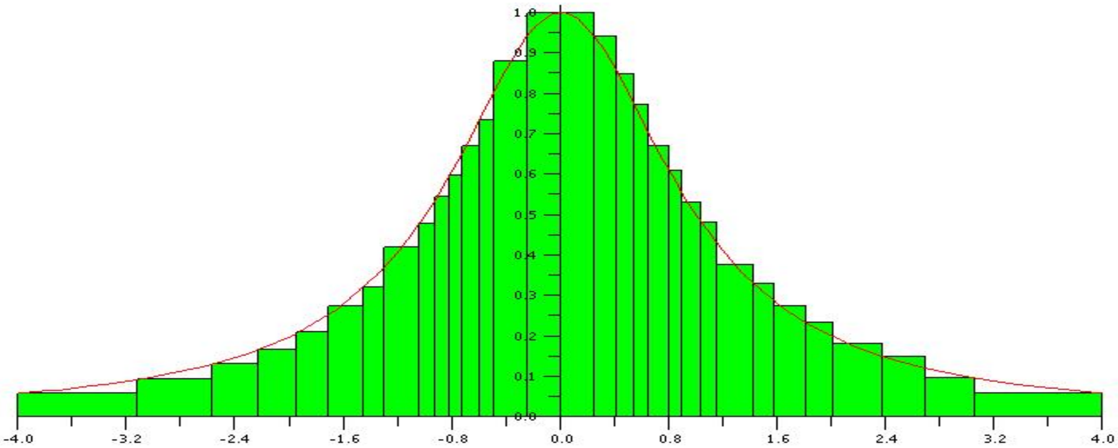

Figure 4: Example 4