SCIE1000, Tutorial Week 13: previous exam questions.

- This tutorial sheet contains two questions covering systems of differential equations, from previous SCIE1000 exams. (Note that the final exam was not open-book in 2009, but was in 2010.)

- Take the exam seriously. Many people find open-book exams to be much harder than closed-book exams, because there are no easy "memorisation marks". You will need to be quite familiar with the approaches we have covered throughout the course, and be able to think about problems and situations that you haven't encountered before.

- Also, the exam is quite long. You will need to work quickly, make rapid decisions about models, justify your choices, interpret answers and draw conclusions. The midsemester exam givesa good idea of what to expect.

1 Questions

1. (Final exam, 2010. Worth 28 marks, so about 28 minutes to work.) Researchers are trialling a new treatment for a form of cancer. Each month:

   - 60 people are newly diagnosed with cancer
   - a person with cancer will: move into remission (30% likelihood); die (20%); or continue with cancer (and treatment) for another month (50%).
   - a person in remission will: have their cancer redevelop (10% likelihood); die (5%); or continue in remission (85%).

   Researchers wish to develop a model based on a system of differential equations.

   (a) Draw a life-cycle diagram with two stages, C representing people with cancer and R representing people in remission.

   (b) Let $C(t)$ and $R(t)$ be the number of people with cancer and in remission at any time t in months. Explain **briefly** why the following pair of differential equations models the situation described above.

   $$
   \begin{aligned}
   C' &= 60 - 0.5C + 0.1R \\
   R' &= 0.3C - 0.15R
   \end{aligned}
   $$

   (c) Find all *stable solutions* to the equations in Part (b), and briefly interpret your answer.

   (d) Assume that at some time T, the numbers of people with cancer and in remission form a stable solution. How many people in total will die in the next 100 months (that is, between times T and $T + 100$)? At time $T + 100$, how many people will have cancer and how many will be in remission? Briefly explain your answers.

   (e) At time $t = 0$, 300 people have cancer and 40 more are in remission. Apply **one** step of Euler's method with a **step size of 0.5 month** to predict the values of **both** C and R after 0.5 month. Show all work.

   (f) Euler's method was applied for a period of 24 months, producing the following graph.

Number of people with cancer and in remission

R(t)

C(t)

Number of people

Time (months)
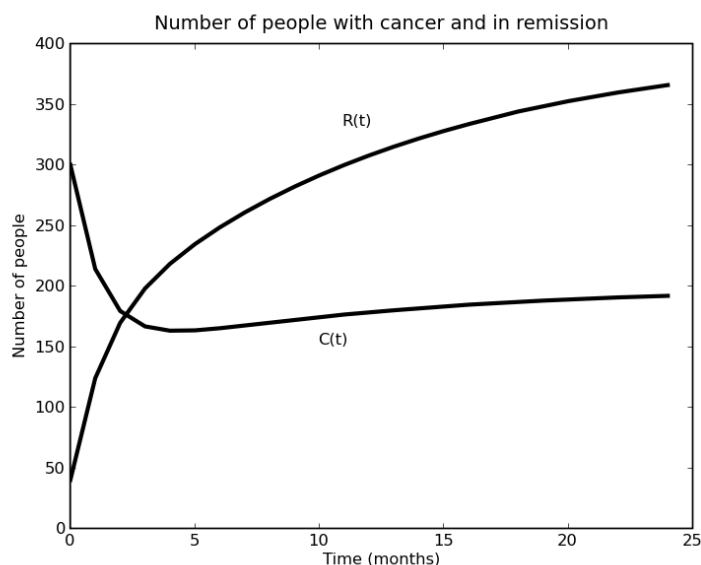
(i) In order to plan for providing ongoing health support, the government seeks your advice on the likely long-term number of people in each stage. What advice would you provide? Justify your answer briefly.

(ii) A biologist proposes modelling $R(t)$ in the long term by a power function, with the power between 0 and 1. Briefly discuss the effectiveness of this approach.

2. (Final exam, 2009. Worth 17 marks, so about 17 minutes to work.) Consider the following system of two differential equations that are modelling two linked phenomena $X(t)$ and $Y(t)$, where t is measured in weeks.

$$X' = -\frac{4\pi}{10}Y \qquad\qquad Y' = \frac{\pi}{10}X$$

(a) If $X(0) = 0$ and $Y(0) = -2$, apply **one** step of Euler's method with a step size of 0.1 week to predict the values of X and Y at $t = 0.1$ weeks. Show all work.

(b) A pair of values (X_s, Y_s) is said to form a *stable solution* to this system of equations if $X' = 0$ and $Y' = 0$ when $X = X_s$ and $Y = Y_s$. Find all such values of (X_s, Y_s). **Interpret your answer.**

(c) Show that

$$X(t) \;=\; 2A\sin\left(\frac{2\pi t}{10}\right)$$

$$Y(t) \;=\; -A\cos\left(\frac{2\pi t}{10}\right)$$

form a solution to the given system of differential equations, where A is a constant. (Hint: if $g(t) = a\sin bt$, where a and b are constants, then $g'(t) = ab\cos bt$. Similarly, if $h(t) = a\cos bt$ then $h'(t) = -ab\sin bt$.)

(d) If $X(0) = 0$ and $Y(0) = -2$, find the value of the constant A.

(e) Use your result from Parts (c) and (d) to sketch $X(t)$ for t between 0 and 20 weeks, showing the correct amplitude, frequency and equilibrium value. (There is no need to sketch $Y(t)$.)

The end

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