Asymptotic Localisation of Neutrinos in Relativistic Quantum Mechanics

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Abstract

In this thesis, we consider the localisation of the neutrino in relativistic quantum mechanics. Aspects of Newton-Wigner localisation of elementary systems are discussed, with their conclusion that localised states cannot be constructed for the 2-component neutrino being particular relevant to us. We prove that positive energy states of the Dirac position operator $\mathbf{x}$ can be constructed, so that these states have arbitrarily small variance in position. We then outline a new concept of localisation, "asymptotic localisation", which we feel offers a reasonable solution to the problem of localising a 2-component neutrino.

Chapter 5

Localisation in Elementary Systems

The notion of the position of a particle has been the subject of much discussion through the evolution of quantum mechanics. In non-relativistic quantum mechanics, with the advent of the Hilbert space theory, the statistical interpretation of measurement and the discovery of Heisenberg's uncertainty relations, position has been given a well-defined meaning in terms of the operator $\mathbf{x}$. It is well-established that we can define generalised eigenstates of $\mathbf{x}$, about the spatial coordinate $\mathbf{a}$ of the form

$$\psi(\mathbf{x}) \;=\; \psi_0 \delta^{(3)}(\mathbf{x}-\mathbf{a})\,, \tag{5.1}$$

where ψ_0 is a constant. Admittedly these states are only the limits of physical states, because they cannot be normalised and hence, are not in $\mathcal{H}$, however, we can approximate them by normalised elements of Hilbert space, whose variance in position is arbitrarily close to zero. Moreover, we can make the region on which these states are non-zero to be as small as we like.

As an example, consider a free particle in one dimensional-space, $\langle x \rangle = 0$, and define the particle's wave function to be

$$\Psi(x) = \begin{cases} \frac{1}{\sqrt{a}} & 0 \leq |x| \leq a, \\ 0 & |x| > a \end{cases},$$

where a is a positive constant. Now Ψ is a normalised element of Hilbert space,

since

$$||\Psi||^2 = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx$$
$$= \int_{-a}^{a} \frac{1}{2a} dx = 1,$$

and the expectation value of x in this state is

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x)|^2 dx$$
$$= \int_{-a}^{a} \frac{x}{2a} dx = \frac{a^2 - a^2}{4a} = 0.$$

Thus Ψ forms a perfectly legitimate state vector for a free particle in one dimension, with uniform probability density $|\Psi(x)|^2 = 1/2a$ on $[-a, a]$, while the probability of the particle being found outside $[-a, a]$ is zero. Thus we can say that the particle is *strictly localised* on $[-a, a]$.

5.1 Localisation for relativistic particles

In relativistic quantum mechanics the status of position is less clear, and a completely satisfactory means of constructing localised states for a single particle has not yet been reached. In this thesis, we are particularly interested in the case of the neutrino, however we shall initially discuss more general approaches, concerning elementary systems. Elementary systems provide an elegant method of describing the states of a relativistic system, however they do not explicitly prescribe a position operator. The only physical quantities directly described are the ten generators of the Poincaré group, namely the momentum-energy 4 vector, the 3 generators of Lorentz boosts and the 3 generators of rotations. Thus the choice of position operator is far from clear. The natural option, in a coordinate representation, would be to choose the operator $\mathbf{x}$. However there are significant difficulties with this interpretation. Firstly, recall that the wave equations associated with relativistic particles admit of both positive and negative energy solutions, which split Hilbert space into positive and negative energy subspaces. The operator $\mathbf{x}$ acts on the whole of Hilbert space and typically cannot be diagonalised on the positive energy subspace alone.

Thus generalised eigenstates of $\mathbf{x}$ would naturally include contributions from negative energy states. This is a troubling proposition because it suggests that a particle initially prepared in a positive energy state can evolve into a state with negative energy, as a result of a position measurement.

This is not the only concern with the use of $\mathbf{x}$ as the position operator. If we consider the neutrino specifically, we know that the Hamiltonian is given by $H = c\,\boldsymbol{\sigma}\cdot\mathbf{p}$. If we wish to calculate the velocity of the particle, we must consider the time-derivative of position, which we defined in chapter 2. Now

$$\begin{aligned}
\mathbf{v} = \dot{\mathbf{x}} &= -\frac{i}{\hbar}[\mathbf{x}, c\boldsymbol{\sigma}\cdot\mathbf{p}] \\
v_i &= -\frac{ic}{\hbar}\sigma_j\left[x_i, p_j\right], \qquad \text{sum over } j \\
&= c\sigma_i, \quad \text{using equation (2.5)}.
\end{aligned}$$

Thus we have the velocity operator $\mathbf{v} = c\boldsymbol{\sigma}$. While $\mathbf{v}$ is Hermitian, the individual components of this velocity operator do not commute with one another, therefore simultaneous measurement of more than one component is not possible. Secondly, the eigenvalues of each component v_i are equal to $\pm c$. Now the neutrino is postulated to travel at the speed of light, so this may seem consistent, however if we now consider the operator $v^2 = 3c^2 I_2$, which commutes with all 3 components of velocity, and would be expected to represent the square of the speed, we can see its eigenvalues are $3c^2$, implying the particle's speed would exceed that of light. With such undesirable properties, we can not consider this velocity operator to be an observable quantity. In fact, it is also clear that $\ddot{\mathbf{x}} \neq \mathbf{0}$, clearly not what one would expect from a free particle, however, once again this is an unobservable quantity.

There is another argument, based on the indeterminacy principle, which claims that any localisation much within the Compton wavelength of a relativistic particle will necessarily result in pair production, rendering the one particle picture invalid. The crux of this argument is that if the particle is localised within its Compton wavelength, the uncertainty in energy may be greater than $2mc^2$, the rest energy of a particle, anti-particle pair. However, within a relativistic one-particle theory, there

exist positive- energy states with arbitrarily large uncertainty in energy, because the energy is unbounded above. Thus if the uncertainty in position is small enough for the uncertainty in energy to be greater than the rest energy of a particle, anti-particle pair, this does not necessarily imply a breakdown of the one-particle picture.

Some theorists have claimed that the very concept of position is not meaningful in a relativistic single particle theory, but this is perhaps connected with the intractable nature of the problem of determining the position operator and its corresponding eigenstates, not from any underlying physical cause. Others have been left unconvinced by these arguments and have sought to determine position operators and their eigenstates with respect to certain desirable properties of the operator or its localised states. These approaches can be broadly grouped along certain lines of research. The problem is essentially to find:

1. An operator $\mathbf{q}$ corresponding to position, and its associated eigenstates (referred to as localised states).

2. The properties of this operator and its localised states.

3. The properties of variables related to position, important examples of these being time and velocity.

Kálnay [9] provides a good review of the methods that have been employed in the search for a solution. Despite this interest in the position operator, there is as yet no consensus on a solution to the problem.

Here we shall concentrate on the approach taken by Newton and Wigner [11] to the localisation problem in their seminal paper. To quote Kálnay [9], "We regard the classical paper by Newton and Wigner (NW) as the most fundamental one on the localisation problem". The Newton-Wigner approach provides perhaps the most well-known solution to the problem. We use the expression "solution" loosely here, since despite the strengths of their paper, the localised states found do not satisfy Einstein causality (Einstein causality states that there can be no propagation faster than the speed of light) and so cannot easily be accepted in a relativistic theory.

5.2 Newton-Wigner Localisation

Newton and Wigner approached the problem of constructing localised states from the perspective of symmetry transformations. For simplicity, they considered only elementary systems, that is dynamical systems whose states form an irreducible representation of the inhomogeneous Lorentz group. To ensure the localised states had reasonable properties, they postulated that such states should satisfy the following:

1. The localized states form a linear space S_0 (so a superposition of two localized states is again such a state);

2. S_0 is invariant under spatial rotations about the origin and reflections in space and time;

3. A spatial displacement of a state in S_0 makes it orthogonal to all states in S_0;

4. Certain mathematical smoothness conditions hold.

These postulates perhaps need some explanation as to their implications. Postulate (1) regarding the space S_0 of localised states is quite clear, being a statement of compatibility of the localised states. Newton and Wigner regarded postulate (3) as the most important. The authors were looking for point-localised states, i.e. the position of the system is localised at a single spatial point, rather than over a volume of space, thus the probability of the particle being at any other spatial point should be zero. This can be ensured by making states localised at **a** orthogonal to their translates to a distinct point **b**. The space S_0 is really the limiting case of a linear space in Hilbert space, as the generalised eigenfunctions we are dealing are only the limits of square-integrable functions, and so do not belong to Hilbert space themselves. A more rigorous derivation of the Newton-Wigner results is given by Wightman [16], however apart from a firmer mathematical foundation, the approach and criticisms remain unchanged.

Newton and Wigner then looked for localised states which satisfied these conditions, from which they could determine the form of the position operator. They

considered separately the case of non-zero and zero spin particles, with finite rest mass, the former requiring a more sophisticated mathematical approach. They concluded that localised states could be found for these systems and they corresponded to a position operator $\mathbf{q}$, which leaves invariant the positive energy subspace of Hilbert space. For the massive Dirac particle, the position operator is identical to that found earlier by Pryce [[13]], and also Foldy and Wouthuysen [[5]]:

$$\tilde{\mathbf{q}} = \mathbf{x} + \frac{\hbar}{2}\left[\frac{c}{E}i\beta\boldsymbol{\alpha} + \frac{c^2}{E(E+mc^2)}\mathbf{p}\times\boldsymbol{\Sigma} - \frac{c^3}{E^2(E+mc^2)}i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}\right], \qquad (5.2)$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Newton and Wigner also concluded that for particles with zero rest mass, localised states were possible for spin 0 and spin $\frac{1}{2}$ particles, however, such states did not exist for higher spin particles. These conclusions were quite a step forward at the time, since the localised states satisfy quite general conditions and the associated position operator acts only on positive energy states

While localised states of a the Newton-Wigner operator can be found for a spin $\frac{1}{2}$ Dirac particle with zero rest mass, this does not apply to the two-component neutrino. The 2-component neutrino has a definite helicity, and this property causes the neutrino to violate the condition of invariance under spatial reflections.

5.3 Difficulties with Newton-Wigner localisation

Unfortunately, the Newton-Wigner localised states are not well-behaved under the action of the inhomogeneous Lorentz group, because problems arise with time evolution. If we consider a particle localised at a point $\mathbf{x}_0$ in 3-space at time t, then at a time $t+T$, special relativity imposes the condition that the particle should remain localised within a sphere of radius $R = cT$, centred at $\mathbf{x}_0$. However, the Newton-Wigner states delocalise instantaneously and as a consequence, fail to satisfy this condition. After this was discovered, it was hoped that by weakening postulate (3)

to orthogonality outside some small volume V, Einstein causality could be satisfied. Physically, this interpretation appears reasonable, as it is likely that particles are not point-like but in fact take up a small volume of space.

It has been shown [6], however, that even with this condition, the localised states will violate Einstein causality at some future time. Hegerfeldt [7], has extended this result by showing that even if the localised states have tails which decay exponentially outside of some region R, Einstein causality will still be violated at later times. However, implicit in Hegerfeldt's work is the assumption that the position operator acts to leave invariant the positive energy part of Hilbert space.

5.4 Localisation for the neutrino

If we now turn our attention to the neutrino, we know that the position operator $\mathbf{x}$ acts on the whole of Hilbert space. This property makes the operator unappealing, because its action leaves neither the positive nor the negative energy subspace invariant. Nevertheless, there is some justification for regarding $\mathbf{x}$ to be the correct position operator. As we noted before, the 2-component neutrino can be described by Dirac's equation (with some constraints imposed). If we consider a charged Dirac particle, such as an electron, moving in an external electro-magnetic field, then we minimally couple the field to the Dirac operator $\mathbf{x}$, not the Newton-Wigner operator $\tilde{\mathbf{q}}$. Additionally, experimental observations have not revealed any extended charge structure to the electron, which is something of a paradox if we cannot, in theory, localise the charge.

If we assume that the position operator for the neutrino is the Dirac position $\mathbf{x}$, then it is still possible to construct 2-spinor states ψ that have contributions only from the positive energy subspace. The mean and variance of $\mathbf{x}$, in these states are well-defined, so we considered it might be advantageous to to find the greatest lower bound on the variance in position, which we will denote $\Delta_{\mathbf{x}}$. A similar investigation [10] has been performed on the Dirac particle, with the surprising result that the greatest lower bound on $\Delta_{\mathbf{x}}$ is in fact 0. This allows one to find positive energy

states of the Dirac particle, whose variance is arbitrarily near to 0. We shall refer to these states as "asymptotically localised", since in terms of Δ_x, the position is, in an intuitive sense, asymptotically close to the mean value. We would like to extend this result to the neutrino, firstly because its mathematical description is very similar to the Dirac particle, and secondly because Newton-Wigner localisation is impossible for the 2-component neutrino. From this perspective, it seems worthwhile to try to find an alternative form of localisation which is acceptable for the 2-component neutrino.

Chapter 6

Arbitrarily Precise Localisation

In this chapter, we derive the result that the lower bound on the variance Δ_x is 0. It suits our purposes to work in the momentum representation: recall from Chapter 2 that a 2-spinor wave function $\psi(\mathbf{x})$ is related to the momentum representation $\varphi(\mathbf{p})$ by the Fourier transform:

$$\varphi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbb{R}^2} \psi(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}\, d^3x\,. \tag{6.1}$$

We seek states $\psi(\mathbf{x})$ of minimum variance, subject to the positive energy condition,

$$H\psi \;=\; +E(\mathbf{p})\psi, \tag{6.2}$$

where $H = c\boldsymbol{\sigma}\cdot\mathbf{p}$, and $E(\mathbf{p}) = c|\mathbf{p}|$. These states must also satisfy the normalisation condition

$$(\psi,\, \psi) \;=\; 1\,. \tag{6.3}$$

Since the variance is dependent on the mean, we need to define the particle's mean position. For simplicity, we shall say that the particle has its mean at the origin. Thus $\Delta_x = (\psi, \mathbf{x}^2\psi)$ and we have the extra condition

$$(\psi, \mathbf{x}\psi) \;=\; 0\,, \tag{6.4}$$

to satisfy. As we wish to work in momentum space, we need to transform our respective conditions as well. We note that in momentum space, the position operator

$\mathbf{x} \to i\hbar\nabla$. The momentum space form of the variance is then

$$\langle \mathbf{x}^2 \rangle = -\hbar^2 \int \varphi(\mathbf{p})^\dagger \nabla^2 \varphi(\mathbf{p})\, d^3p\,. \tag{6.5}$$

We can integrate () by parts, so that it takes the form:

$$\langle \mathbf{x}^2 \rangle = \hbar^2 \int \frac{\partial \varphi^\dagger}{\partial \mathbf{p}} \cdot \frac{\partial \varphi}{\partial \mathbf{p}}\, d^3p\,. \tag{6.6}$$

We now introduce the notation $\partial\varphi/\partial p^i = \varphi_{,i}$, using the summation convention on repeated indices. We wish to minimize

$$I = \int \varphi^\dagger_{,i} \varphi_{,i}\, d^3p\,, \tag{6.7}$$

subject to

$$\int \varphi^\dagger \varphi_{,j}\, d^3p = 0\,, \quad j = 1,2,3\,, \tag{6.8}$$

$$\int \varphi^\dagger \varphi\, d^3p = 1\,, \tag{6.9}$$

and the positive-energy condition

$$H\varphi = +E(\mathbf{p})\varphi \tag{6.10}$$

We now define energy basis 2-spinors $u_a(\mathbf{p})\,, a = 1,2$ satisfying

$$Hu_1(\mathbf{p}) = +E(\mathbf{p})u_1(\mathbf{p})\,, \tag{6.11}$$

$$Hu_2(\mathbf{p}) = -E(\mathbf{p})u_2(\mathbf{p})\,, \tag{6.12}$$

$$u_a^\dagger(\mathbf{p})u_b(\mathbf{p}) = \delta_{ab}\,. \tag{6.13}$$

The construction of such 2-spinors is well-known [14]. we can satisfy (6.11-6.13) by setting

$$u_a = \frac{1}{\mathcal{E}}(|\mathbf{p}|I_2 + A\sigma_3)e_a \tag{6.14}$$

where

$$\mathcal{E} = \sqrt{|\mathbf{p}|(|\mathbf{p}| + p_3)}\,,$$

$$A = \boldsymbol{\sigma} \cdot \mathbf{p}\,,$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general 2-spinor momentum-space state $\varphi(\mathbf{p})$ can now be expanded as

$$\varphi(\mathbf{p}) = f_a(\mathbf{p}) u_a(\mathbf{p})\,, \tag{6.15}$$

where $f_a\,, a = 1, 2$, are complex-valued, square-integrable, scalar functions, and we also use the summation convention over a-values.

From (6.15) we have

$$\varphi_{,i} = f_{a,i} u_a + f_a u_{a,i}\,, \tag{6.16}$$

and we can write

$$u_{a,i} = Q_{iab} u_b \tag{6.17}$$

for suitable coefficients $Q_{iab}(\mathbf{p})$, as determined in Appendix A, so that

$$\varphi_{,i} = [f_{a,i}\delta_{ab} + f_a Q_{iab}] u_b\,. \tag{6.18}$$

With the help of (6.13) the integrands of (6.7, 6.8, 6.9) then reduce to

$$\varphi^\dagger_{,i}\varphi_{,i} = f^*_{a,i} f_{a,i} + f_a Q_{iab} f^*_{b,i} + f^*_a Q^*_{iab} f_{b,i} + f_a Q_{iac} Q^*_{ibc} f^*_b\,, \tag{6.19}$$

$$\varphi^\dagger \varphi_{,j} = f^*_a f_{a,j} + f_a Q_{jab} f^*_b\,, \tag{6.20}$$

$$\varphi^\dagger \varphi = f^*_a f_a\,, \tag{6.21}$$

respectively, where the summation convention applies to all repeated indices. The constraint (6.10) now reduces to the condition

$$f_2 = 0\,, \tag{6.22}$$

so that the sums over a and b (but not c) in (6.19 - 6.21) are now restricted to the value 1. Noting from equation (7.9) that $Q_{i11}(\mathbf{p})$ is an odd-function of $\mathbf{p}$ we see

that a *sufficient* condition to satisfy (6.8) is to make (6.20) an odd-function of $\mathbf{p}$ by choosing f_1 to be a real-valued, even-function of p^1, p^2 and p^3. Then (6.9) reduces to

$$\int f_1^2 \, d^3p = 1 \,, \tag{6.23}$$

and (6.7) to

$$I = \int \left[F_{,i}^T F_{,i} + F^T Q_i F_{,i} + F_{,i}^T Q_i^\dagger F + F^T Q_i Q_i^\dagger F \right] d^3p \,, \tag{6.24}$$

where $F = (f_1, 0)^T$ and Q_i is the 2×2 matrix whose ab-th element is Q_{iab}. From from the appendix, we have that the elements Q_{i11} satisfy the relation

$$Q_{i11}^* = -Q_{i11} \,.$$

Hence, we have

$$\begin{aligned} F^T Q_i F_{,i} + F_{,i}^T Q_i^\dagger F &= f_i Q_{i11} \frac{\partial f_1}{\partial p^i} + \frac{\partial f_1}{\partial p^i} Q_{i11}^* f_1 & (6.25) \\ &= \frac{\partial f_1}{\partial p^i} f_i (Q_{i11} - Q_{i11}) = 0 \,. & (6.26) \end{aligned}$$

Thus these terms make no contribution to the integral (6.24).

Substituting from equation (7.10) in the fourth term, we then get

$$I = \int \left[\left(\frac{\partial f_1(\mathbf{p})}{\partial p_i} \right)^2 + f_1(\mathbf{p})^2 R(\mathbf{p}) \right] d^3p \,, \tag{6.27}$$

where

$$R(\mathbf{p}) = \frac{(3|\mathbf{p}| + p_3)(|\mathbf{p}| + p_3)}{\mathcal{E}^4} \,. \tag{6.28}$$

We are now in a position to show that 0 is the greatest lower bound on I in (6.27), subject to (6.23), where f_1 is a real-valued, even-function of p^1, p^2 and p^3. Since $\Delta_{\mathbf{x}} \geq 0$, it will then follow that 0 is the greatest lower bound on $\Delta_{\mathbf{x}}$ with $\langle \mathbf{x} \rangle = \mathbf{0}$. To show this, suppose that a real $f_1(\mathbf{p})$ has been chosen, even in p^1, p^2 and p^3, satisfying (6.23), and leading to some value I on substitution in (6.27). For the purposes of the proof, we shall suppose further that f_1 is continuously differentiable

on all of momentum-space, and is sufficiently well-behaved at $\mathbf{p} = \mathbf{0}$ and at infinity to guarantee the convergence of all integrals involved. Then let

$$\mathbf{q} = \beta\mathbf{p}\,, \quad 0 < \beta < 1 \tag{6.29}$$

and set

$$\tilde{f}_1(\mathbf{p}) = \sqrt{\beta^3} f_1(\beta\mathbf{p}) = \sqrt{\beta^3} f_1(\mathbf{q})\,. \tag{6.30}$$

It then follows that

$$\frac{\partial \tilde{f}_1}{\partial p^i} = \sqrt{\beta^3}\frac{\partial f_1}{\partial q^j}\frac{\partial q^j}{\partial p^i} = \sqrt{\beta^5}\frac{\partial f_1}{\partial q^i}\,. \tag{6.31}$$

From (6.30) we have

$$\int \tilde{f}_1(\mathbf{p})^2 d^3p = \beta^3 \int f_1\mathbf{q})^2 \frac{d^3q}{\beta^3} = 1\,, \tag{6.32}$$

so that $\tilde{f}_1$ satisfies all the conditions imposed on f_1 and f_2. We can substitute $\tilde{f}_1$ in place of f_1 in (6.27), which leads with the help of (6.30) and (6.31) to a new value

$$\begin{aligned}\tilde{I} &= \int \left[\left(\frac{\partial \tilde{f}_1(\mathbf{p})}{\partial p_i}\right)^2 + \tilde{f}_1(\mathbf{p})^2)\frac{(3|\mathbf{p}| + p_3)(|\mathbf{p}| + p_3)}{\mathcal{E}^4}\right] d^3p\,, \\ &= \beta^5 \int \left[\left(\frac{\partial f_1(\mathbf{q})}{\partial q_i}\right)^2 + f_1(\mathbf{q})^2)\frac{(3|\mathbf{q}| + q_3)(|\mathbf{q}| + p_3)}{\mathcal{E}^4}\right] \frac{d^3q}{\beta^3}\,,\end{aligned}$$

Thus,

$$\tilde{I} = \beta^2 \int \left[\left(\frac{\partial f_1(\mathbf{q})}{\partial q_i}\right)^2 + f_1(\mathbf{q})^2)\frac{(3|\mathbf{q}| + q_3)(|\mathbf{q}| + p_3)}{\mathcal{E}^4}\right] d^3q\,. \tag{6.33}$$

Clearly as $\beta \to 0^+$ this integral goes to zero. Hence, the greatest lower bound on the variance of position is zero.

6.1 Asymptotically localising sequences

As a result of this, at any point $\mathbf{a}$, we can construct a sequence $\{\psi_n\}_{n=1}^{\infty}$ of positive energy states, each with expectation value of $\mathbf{x}$ equal to $\mathbf{a}$, whose corresponding sequence of variances Δ_x converges to 0. Moreover, the associated sequence of densities $\rho_n(\mathbf{x}) = \psi_n(\mathbf{x})^\dagger\psi_n(\mathbf{x})$ approaches $\delta^{(3)}(\mathbf{x}-\mathbf{a})$. We could consider such a sequence

to be arise as a result of a series of measurements of position, each measurement having greater precision. We shall refer to these sequences as "asymptotically localising sequences". We add that the states which comprise these sequences, have "tails" extending to infinity (much like the tails of a Gaussian distribution). Thus the neutrinos they describe have position which cannot be completely localised within any region of space.

The formulation of the the above proof suggests a method of constructing an asymptotically localising sequence (ALS) at $\mathbf{0}$. We choose f_1 to be a real-valued, even function of p_1, p_2 and p_3, and form the sequence $\{\varphi_n(\mathbf{p})\}_{n=1}^{\infty}$, with

$$\varphi_n(\mathbf{p}) = \frac{1}{n^{3/2}} f_1\left(\frac{\mathbf{p}}{n}\right) u_1(\mathbf{p}),$$

where $1/n$ assumes the role of β in the above proof. Hence, the associated sequence of variances $\{\Delta_x\}$ is monotonic decreasing, and bounded below by 0. We can obtain an ALS at a general point $\mathbf{a}$, from the translation

$$\varphi_n(\mathbf{p}) \rightarrow e^{-i\mathbf{a}\cdot\mathbf{p}/\hbar}\varphi_n(\mathbf{p}),\ \psi_n(\mathbf{x}) \rightarrow \psi_n(\mathbf{x}-\mathbf{a}).$$

As an example of an ALS at 0 , we can choose a real, even, square-integrable function

$$f(\mathbf{p}) = \frac{1}{(2\pi\alpha)^{3/2}} e^{-\mathbf{p}^2/2\alpha},$$

so that

$$\varphi_n(\mathbf{p}) = \frac{1}{(2\pi\alpha n^2)^{3/2}} e^{-\mathbf{p}^2/2\alpha n^2} u_1(\mathbf{p}).$$

As a point of comparison with the Newton-Wigner localised states, it is interesting to examine the properties of ALS's with respect to the Newton-Wigner postulates. Summarising the results of Bracken and Melloy [[10]], we find that ALS's behave as follows:

- The effect of a rotation, or a time reversal operation, applied to an ALS at 0 is to produce another such ALS. However, since neutrino states are not invariant under spatial reflections, we cannot satisfy all the conditions of postulate (2).

- A spatial translation of an ALS at **a** will produce another ALS at the appropriately translated point **b**.

- In general, the elements of an ALS will not be orthogonal to their translates in space, however the scalar product of successive elements of an ALS at **a** with successive elements of an ALS at a distinct point **b** will converge to zero.

- The superposition of two ALS's at **a** will not in general produce another ALS at **a**.

It is not surprising that asymptotically localised fail to satisfy these last two properties. As a result of their infinite tails such states cannot be strictly localised within any region, and are thus unlikely to be orthogonal. Secondly, two ALS's localised about a point **a** will typically have their peaks about slightly different points, hence the difference of two such states may be zero near **a** and have peaks at some distance form **a**.

Thus, in the light that they form approximate, rather than point-, localised states we may conclude that ALS's behave reasonably with respect to the Newton-Wigner postulates,

Chapter 7

Conclusions

We feel that asymptotic localisation provides a reasonable definition of localised states and should be adopted as *the* concept of localisation for the 2-component neutrino. As we have emphasised, Newton-Wigner localisation cannot be applied to the 2-component neutrino, and we believe that asymptotic localisation adequately fills this void.

To further emphasise the suitability of asymptotic localisation to the neutrino, note that we can define a current density $\mathbf{j}(\mathbf{x},t)$ associated with the probability density $\rho(\mathbf{x},t)$ such that the two satisfy a conservation equation. The conservation equation for the Dirac current density (for the 2-component neutrino) is given by

$$j_i(\mathbf{x},t) \;=\; c\psi^\dagger(\mathbf{x},t)\sigma_i\psi(\mathbf{x},t)\,. \tag{7.1}$$

The conservation equation is important in itself, however, more significant to our concept of localisation is the observation that, at each point in space and time, these quantities satisfy the inequality

$$|\mathbf{j}(\mathbf{x},t)\cdot\mathbf{n}| \;\leq\; c\rho(\mathbf{x},t)\,, \tag{7.2}$$

for every constant unit vector $\mathbf{n}$, because no eigenvalue od $\boldsymbol{\sigma}\cdot\mathbf{n}$ can be greater than 1 in magnitude. This inequality then provides the necessary and sufficient condition for the density ρ to spread at speeds no greater than the speed of light. This is a crucial result, for it implies that the states which form an ALS, will not violate Einstein causality. Along with this, we have that the Dirac position operator demonstrates reasonable properties under Lorentz boosts.

We still have the apparent difficulty that each component of the velocity vector has eigenvalues of $\pm c$, and that it is not a multiple of the momentum of the neutrino. However, we cannot diagonalise the components of velocity in positive energy states, hence these eigenvalues are not observable. This should not be surprising, since the Dirac position operator is not diagonalisable on positive energy states either. Consider, however, that the expectation value of $\dot{\mathbf{x}}$ is the same as that of $c^2 H^{-1}\mathbf{p}$, which is the value that classical considerations lead us to expect. We add, however, that we cannot make the uncertainty in velocity arbitrarily small because

$$\mathrm{v}^2 = c^2\sigma_i\sigma_i = 3c^2 I_2\,,$$

and thus

$$(\varphi, \mathrm{x}^2\varphi) = 3c^2(\varphi, \varphi) = 3c^2\,,$$

if φ is a normalised element of Hilbert space.

With the adoption of asymptotic localisation, the difficulties associated with the Dirac position operator dissipate for the most part, and provided we are willing to accept approximate localisation, we believe that this provides a reasonable concept of localisation of the neutrino. We reiterate that the elements of an ALS have tails that extend to infinity, so that the probability of the particle being at any spatial point is non-zero. The behaviour of these tails needs some further investigation, so we can determine what sort of functions comprise an ALS. Finally, it is evidently of interest to see if asymptotic localisation can in be applied in some form to the photon, which also lacks Newton-Wigner localised states.

Appendix

Consider the operator $A = \boldsymbol{\sigma} \cdot \mathbf{p}$. We now define 2-spinors u_a, $a = 1, 2$ as follows:

$$u_a = \frac{1}{\mathcal{E}}(|\mathbf{p}|I_2 + A\sigma_3)e_a = Ue_a, \tag{7.3}$$

$$\text{where } \mathcal{E} = \sqrt{2|\mathbf{p}|\,(|\mathbf{p}| + p_3)}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These u_a are normalized vectors since they satisfy the relations $u_a^\dagger u_b = \delta_{ab}$. We need to find the partial derivatives of u_a with respect to the components of momentum. Let us define

$$\frac{\partial u_a}{\partial p_i} = \frac{\partial U}{\partial p_i} = Q_{iab}u_b, \tag{7.4}$$

where the summation convention is employed on the bs. We need to find the form of the Q_{iab}, so if we now multiply on the left by $u_c^\dagger$, we have

$$\begin{aligned} u_c^\dagger \frac{\partial U}{\partial p_i} e_a &= Q_{iab}\, u_c^\dagger u_b \\ &= Q_{iac} \end{aligned}$$

This implies that

$$Q_{iac} = e_c^T U^\dagger \frac{\partial U}{\partial p_i} e_a. \tag{7.5}$$

So we can let Q_i be the 2×2 matrix with Q_{iac} being the element in the a-th row and c-th column, and according to equation (7.4) we have $Q_i^T = U^\dagger \frac{\partial U}{\partial p_i}$. To find the form of Q, we need to evaluate $U^\dagger \frac{\partial U}{\partial p_i}$. Noting that

$$\frac{\partial A}{\partial p_i} = \sigma_i\,,$$

We have

$$\begin{aligned}
\frac{\partial U}{\partial p_i} &= \frac{\partial}{\partial p_i}\left[\frac{1}{\mathcal{E}}(|\mathbf{p}|I_2 + A\sigma_3)\right] \\
&= -\frac{1}{\mathcal{E}^2}\left[2p_i + \frac{p_i p_3}{|\mathbf{p}|} + |\mathbf{p}|\delta_{i3}\right] U + \frac{1}{\mathcal{E}}\left(\frac{p_i}{|\mathbf{p}|}I_2 + \sigma_i\sigma_3)\right). \\
\text{Then } Q_i^T &= -\frac{1}{\mathcal{E}^2}\left[2p_i + \frac{p_i p_3}{|\mathbf{p}|} + |\mathbf{p}|\delta_{i3}\right] I_2 \\
&\quad + \frac{1}{\mathcal{E}^2}\left(|\mathbf{p}|I_2 + A\sigma_3\right)\left(\frac{p_i}{|\mathbf{p}|}I_2 + \sigma_i\sigma_3)\right).
\end{aligned}$$

The form of the Q-matrices is then,

$$Q_1^T = \frac{i}{\mathcal{E}^2}\left[-|\mathbf{p}|\sigma_2 + \frac{p_1^2}{|\mathbf{p}|}\sigma_2 - \frac{p_1 p_2}{|\mathbf{p}|}\sigma_1 - p_2\sigma_3 - p_3\sigma_2\right], \tag{7.6}$$

$$Q_2^T = \frac{i}{\mathcal{E}^2}\left[|\mathbf{p}|\sigma_1 - \frac{p_2^2}{|\mathbf{p}|}\sigma_1 + \frac{p_1 p_2}{|\mathbf{p}|}\sigma_2 + p_1\sigma_3 + p_3\sigma_1\right], \tag{7.7}$$

$$Q_3^T = \frac{i}{\mathcal{E}^2}\left[\frac{p_3 p_1}{|\mathbf{p}|}\sigma_2 - \frac{p_3 p_2}{|\mathbf{p}|}\sigma_1 + p_1\sigma_2 - p_2\sigma_1\right]. \tag{7.8}$$

In particular, we need the elements Q_{i11}, which are

$$Q_{111} = -ip_2\,, \quad Q_{211} = ip_1\,, \quad Q_{311} = 0\,. \tag{7.9}$$

Clearly, these elements are odd functions of $\mathbf{p}$. Also note that $Q^*_{i11} = -Q_{i11}$.

We shall also require $Q_i^* = (Q_i^T)^\dagger$. Since equations (7.6 - 7.8) have a factor of i in all terms, $Q_i^* = -Q_i^T$. We can now calculate the product $Q_i^* Q_i^T = -(Q_i^T)^2$. After some rather tedious algebra, we arrive at the simple relation

$$Q_i^* Q_i^T = \frac{(|\mathbf{p}| + p_3)(3|\mathbf{p}| + p_3)}{\mathcal{E}^4} I_2\,, \tag{7.10}$$

with summation over i implied.

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