To summarize: The phase space rep is defined by the Weyl-Wigner transform:

$$A = \mathcal{W}(\hat{a}) \qquad \hat{a} = \mathcal{W}^{-1}(A)$$

$$\mathcal{T} \xrightarrow{\mathcal{W}} \mathcal{K} \qquad \mathcal{K} \xrightarrow{\mathcal{W}^{-1}} \mathcal{T}$$

In $\mathcal{T}$: $((\hat{a}, \hat{b})) = \mathrm{Tr}(\hat{a}^\dagger \hat{b})$. In $\mathcal{K}$: $(A, B) = \frac{1}{2\pi\hbar} \int A(q,p)^* B(q,p)\, dq\, dp\,.$

$$A \star B = \mathcal{W}(\hat{a}\hat{b}) \neq \mathcal{W}(\hat{b}\hat{a}) = B \star A$$

$$\boxed{A(q,p) = \int a_K(q - y/2, q + y/2)\, e^{ipy/\hbar}\, dy}$$

$$\boxed{a_K(x,y) = \frac{1}{2\pi\hbar} \int A(\tfrac{x+y}{2}, p)\, e^{ip(x-y)/\hbar}\, dp}$$

Ex. $\mathcal{W}\left(e^{i(\alpha\hat{q}-\beta\hat{p})}\right) = e^{i(\alpha q-\beta p)}, \qquad \mathcal{W}^{-1}\left(e^{i(\alpha q-\beta p)}\right) = e^{i(\alpha\hat{q}-\beta\hat{p})}$ (Weyl)

It follows that $\mathcal{W}(\hat{I}) = 1, \quad \mathcal{W}(\hat{p}^n) = p^n, \quad \mathcal{W}(\hat{q}^n) = q^n,$

$$\mathcal{W}^{-1}(q^2p^2) = \frac{1}{6}(\hat{q}^2\hat{p}^2 + \hat{q}\hat{p}\hat{q}\hat{p} + \hat{p}\hat{q}^2\hat{p} + \hat{p}^2\hat{q}^2 + \hat{p}\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}^2\hat{q})$$

$$\mathcal{W}([A\hat{q}^2 + B\hat{p}^2]^2) = [Aq^2 + Bp^2]^2 - 3AB\hbar$$

and so on. (Weyl's ordering rules.)

Lecture 2 The Wigner function

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$$W = \mathcal{W}(\tfrac{1}{2\pi\hbar}\,\hat{\rho})$$

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• We see that

$$\int W(q,p,t)\,dq\,dp = 1\,, \qquad \langle\hat{a}\rangle(t) = \int A(q,p)\,W(q,p,t)\,dq\,dp = \langle A\rangle(t)\,.$$

- Furthermore,

$$\int W(q,p,t)\,dp = \frac{1}{2\pi\hbar}\int \psi(q-y/2,t)\psi(q+y/2,t)^*\, e^{ipy/\hbar}\,dy\,dp = \psi(q,t)^*\psi(q,t)\,,$$

and it is also easily seen that

$$\int W(q,p,t)\,dq = \tilde{\psi}(p,t)^*\tilde{\psi}(p,t)\,.$$

In fact, $\int_L W(q,p,t)\,dq\,dp \geq 0$, where L is any straight line $aq+bp = \mathrm{const.}$ in the phase plane, is the marginal probability density for the variable $a\hat{q}+b\hat{p}$.

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- However, W *cannot* be regarded as a classical probability density.

- Consider $\psi(x,t) = \psi(x) = \mathrm{const.}\, e^{-\alpha x^2/2}$. This gives

$$\begin{aligned} W(q,p) &= \mathrm{const.} \int e^{-\alpha(q-y/2)^2}\, e^{-\alpha(q+y/2)^2}\, e^{ipy/\hbar}\, dy \\ &= \frac{1}{\pi\hbar}\, e^{-(\alpha q^2 + p^2/(\alpha\hbar^2))} \end{aligned}$$

Note how the spread in momentum increases as we make α bigger to decrease the spread in position — the Wigner function 'knows' about $\hbar$ and the uncertainty principle!

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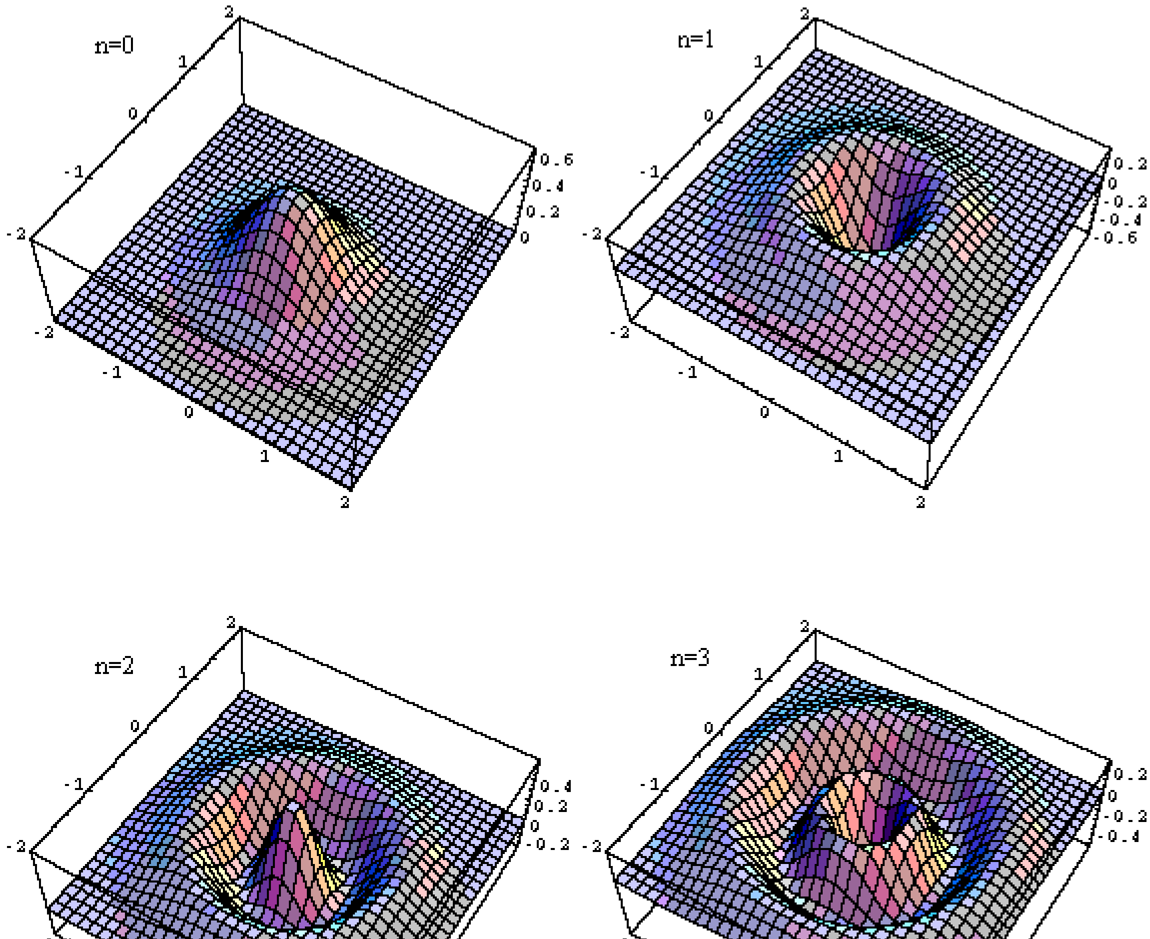
- More striking is to consider something like $\psi(x,t) = \psi(x) = \mathrm{const.}\, xe^{-\alpha x^2/2}$, which gives

$$W(q,p) \;=\; \mathrm{const.} \int (q-y/2)(q+y/2) e^{-\alpha(q-y/2)^2}\, e^{-\alpha(q+y/2)^2}\, e^{ipy/\hbar}\, dy$$

$$=\; \frac{1}{\pi\hbar}\, e^{-(\alpha q^2 + p^2/(\alpha\hbar^2))} \left(2(\alpha q^2 + p^2/(\alpha\hbar^2)) - 1\right) \;.$$

— not everywhere positive! For this reason W is called a *quasiprobability density*.

Note that in this example the area of the ellipse on which W is negative has area $\hbar/2$.


n=0
n=1
n=2
n=3
0.6
0.4
0.2
0
-0.2
-0.4
-0.6

- Other non-classical properties of W include

$$\int W(q,p,t)^2\, dq\, dp \le \frac{1}{2\pi\hbar}\,,$$

with equality only in the case of pure states, and

$$-\frac{1}{\pi\hbar} \le W(q,p,t) \le \frac{1}{\pi\hbar}\,. \qquad (*)$$

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• It is known that the only pure states that give rise to nonnegative Wigner functions are coherent states.

It is not known which mixed states give rise to nonegative Wigner functions . . .

To see where $(*)$ comes from, consider the parity operator on Hilbert space:

$$\hat{P}|x\rangle = |-x\rangle\,,$$

for which

$$P_K(x,y) = \langle x|\hat{P}|y\rangle = \delta(x+y)\,.$$

Its phase space representative is

$$P(q,p) = \int \delta((q-y/2)+(q+y/2))\,e^{ipy/\hbar}\,dy = \pi\hbar\,\delta(q)\,\delta(p)\,.$$

Then

$$\langle\hat{P}\rangle(t) = \int P(q,p)W(q,p,t)\,dq,dp = \pi\hbar\,W(0,0,t)\,.$$

Since $\hat{P}^2 = \hat{I}$, we have

$$-1 \leq \langle\hat{P}\rangle \leq 1$$

and hence

$$-\frac{1}{\pi\hbar} \leq W(0,0,t) \leq \frac{1}{\pi\hbar}\,.$$

- Consider the quasiprobability mass Q_S on a given subregion S of phase space,

$$Q_S = \int_S W(q,p,t)\, dq\, dp\,.$$

It follows from $(*)$ that

$$-\frac{A}{\pi\hbar} \leq Q_S \leq \frac{A}{\pi\hbar}$$

where A is the area of S.

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where A is the area of S.

- Can we find the tightest possible bounds on Q_S, over all possible Wigner functions W?

We rewrite Q_S as

$$Q_S = \int \chi(q,p) W(q,p,t)\, dq\, dp$$

where $\chi(q,p)$ is the *characteristic function* of S. Then we see that

$$Q_S = \langle \hat{\chi} \rangle$$

where $\hat{\chi} = \mathcal{W}^{-1}(\chi)\,.$

It now follows that the greatest lower and least upper bounds on Q_S are the greatest lower and least upper bounds on the spectrum of the operator $\hat{\chi}$ on Hilbert space.

So we are led to consider the eigenvalue problem

$$(\hat{\chi}\,\varphi)(x) \equiv \int \chi_K(x,y)\,\varphi(y)\,dy = \lambda\varphi(x)\,,$$

where

$$\chi_K(x,y) = \frac{1}{2\pi\hbar}\int \chi(\tfrac{x+y}{2},p)\,e^{ip(x-y)/\hbar}\,dp = \frac{1}{2\pi\hbar}\int_{(\frac{x+y}{2},p)\in S} e^{ip(x-y)/\hbar}\,dp\,.$$

Example: Introduce a parameter L with dimensions of length, and change to dimensionless variables

$$q \to q/L\,, \qquad p \to Lp/\hbar\,, \qquad \hbar \to \hbar/\hbar = 1\,, \qquad W \to \hbar W\,, \qquad etc.$$

Consider a region in the phase plane of the form shown in Fig. 1:–

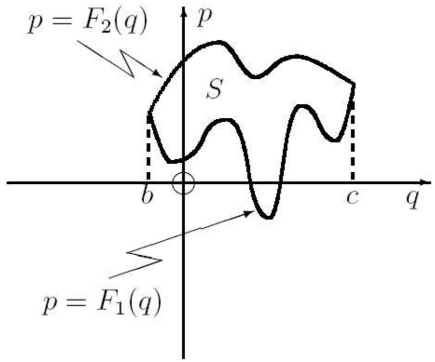

Fig.1. A typical region S in the phase-plane.

For such a region, the characteristic function takes the form

$$\chi(q,p) = \begin{cases} 1 & b < q < c\,, \qquad F_1(q) < p < F_2(q) \\ \\ 0 & \text{otherwise} \end{cases}$$

and the kernel becomes

$$\begin{aligned} \chi_K(x,y) &= \frac{1}{2\pi}\int_{F_1([x+y]/2)}^{F_2([x+y]/2)} e^{ip(x-y)}\,dp \\ &= \frac{e^{i(x-y)F_2([x+y]/2)} - e^{i(x-y)F_1([x+y]/2)}}{2\pi i(x-y)}\,. \end{aligned}$$

for $2b < x+y < 2c$, and 0 otherwise. Note that the singularity at $x = y$ is only apparent.

• Specializing to the case of the disc of radius a centred on the origin, $q^2 + p^2 \leq a^2$, we have

$$F_1(q) = -\sqrt{a^2 - q^2}\,, \qquad F_2(q) = \sqrt{a^2 - q^2}$$

for $-a \leq q \leq a$, and so have to consider the eigenvalue equation

$$\int_{-2a-x}^{2a-x} \frac{\sin[(x-y)\sqrt{a^2 - (x+y)^2/4}]}{\pi(x-y)}\,\varphi(y)\,dy = \lambda\varphi(x)\,.$$

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• From the definitions of χ and $\hat{\chi}$, the nth eigenvalue λ_n is the quasiprobability mass on the disc for the Wigner function $W_n(q,p)$ corresponding to the state with wavefunction $\varphi_n(x)$.

This Wigner function is easily calculated to be

$$W_n(q,p) = (-1)^n \frac{1}{\pi}\, e^{-(q^2+p^2)} L_n[2(q^2 + p^2)]$$

where L_n is the Laguerre polynomial.

Therefore we have

$$\lambda_n \equiv \lambda_n(a) = \int_0^{a^2} L_n(2u)\, e^{-u}\, du \qquad n = 0,\, 1,\, 2,\, \ldots .$$

Thus

$$\lambda_0(a) = 1 - e^{-a^2}\,, \qquad \lambda_1(a) = 1 - (1 + 2a^2)e^{-a^2}\,, \qquad \lambda_2(a) = (1 + 2a^4)e^{-a^2}\,,$$

$$\lambda_3(a) = 1 - (1 + 2a^2 - 2a^4 + \tfrac{4}{3}a^6)e^{-a^2}\,,$$

and so on.

Fig. 2 shows the graphs of $\lambda_n(a)$ *versus* a for $n = 0,\, 1,\, 2,\, 3$, and also the graphs of the greatest and least eigenvalues $\lambda_{max}(a)$ and $\lambda_{min}(a)$ (bold lines).

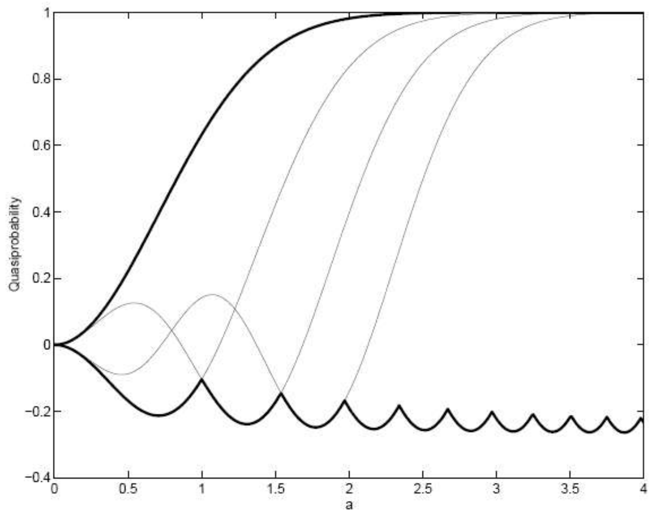

Fig. 2. Left to right: graphs of λ_n for $n = 0, 1, 2, 3$, and also of λ_{max}, λ_{min} (bold lines).

• Note that $\lambda_{max}(a) = \lambda_0(a) = 1 - e^{-a^2}$, whereas the graph of $\lambda_{min}(a)$ has the peculiar scallopped shape shown, because

$\lambda_{min}(a) = \lambda_1(a)$ for $0 \leq a < a_1$,

$\lambda_{min}(a) = \lambda_2(a)$ for $a_1 \leq a < a_2 \quad etc.$,

where

a_1 is the greatest value of a at which $\lambda_1(a) = \lambda_2(a)$,

a_2 is the greatest value of a at which $\lambda_2(a) = \lambda_3(a)\,, \quad etc.$

Thus $a_1 = 1$, $a_2 = \sqrt{(3+\sqrt{3})/2}$, *etc.*

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• The quasiprobability mass on the disc at any value of a must lie between the bold lines at that value of a, whereas a classical probability mass could lie anywhere in $[0, 1]$.

There are implications for quantum tomography ...

The ideas of classical tomography are well-known ...

(pictures from M.G. Raymer, *Contemporary Physics* **38** (1997), 343–355.)

integrals are from $-\infty$ to $+\infty$. The generalization of equation (2) to an arbitrary variable R is

$$\tilde{\psi}_R(R) = \int \psi(x) U(x, R)\, \mathrm{d}x, \tag{3}$$

where $U(x,R)$ is some transformation function. Then the probability density for obtaining values R upon measurements is $\left|\tilde{\psi}_R(R)\right|^2$.

It is not understood deeply *why* the wave functions transform in this way. This question is likely related to the question why the wave function is complex. Again the complementarity idea comes in. Any two variables whose wave function representations transform as in equation (2) are said, first by Niels Bohr, to be complementary [6].

If the state if identified with the set of probability densities for *all* possible observable variables, as in Item (1), then it might be thought that in order to determine a state, you would need to measure the distributions of *all* of the variables and then try to invert the data to find the state. It turns out, though, to our good fortune, that one needs only to measure the distributions of a subset of all possible variables. I call such a subset of variables Tomographically

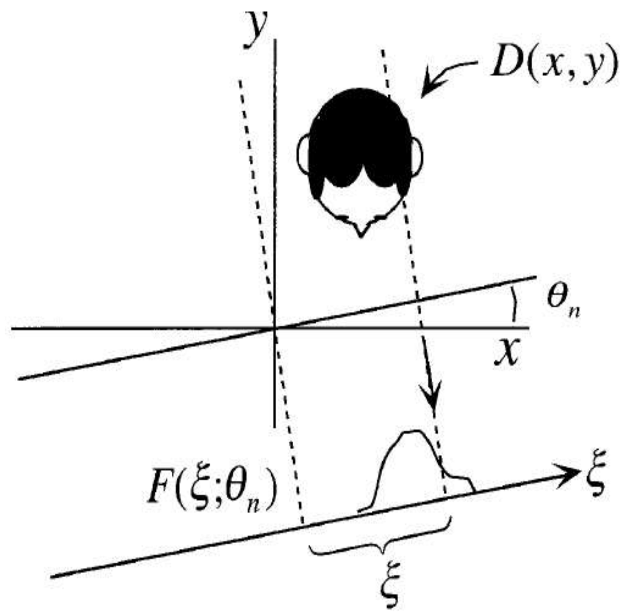
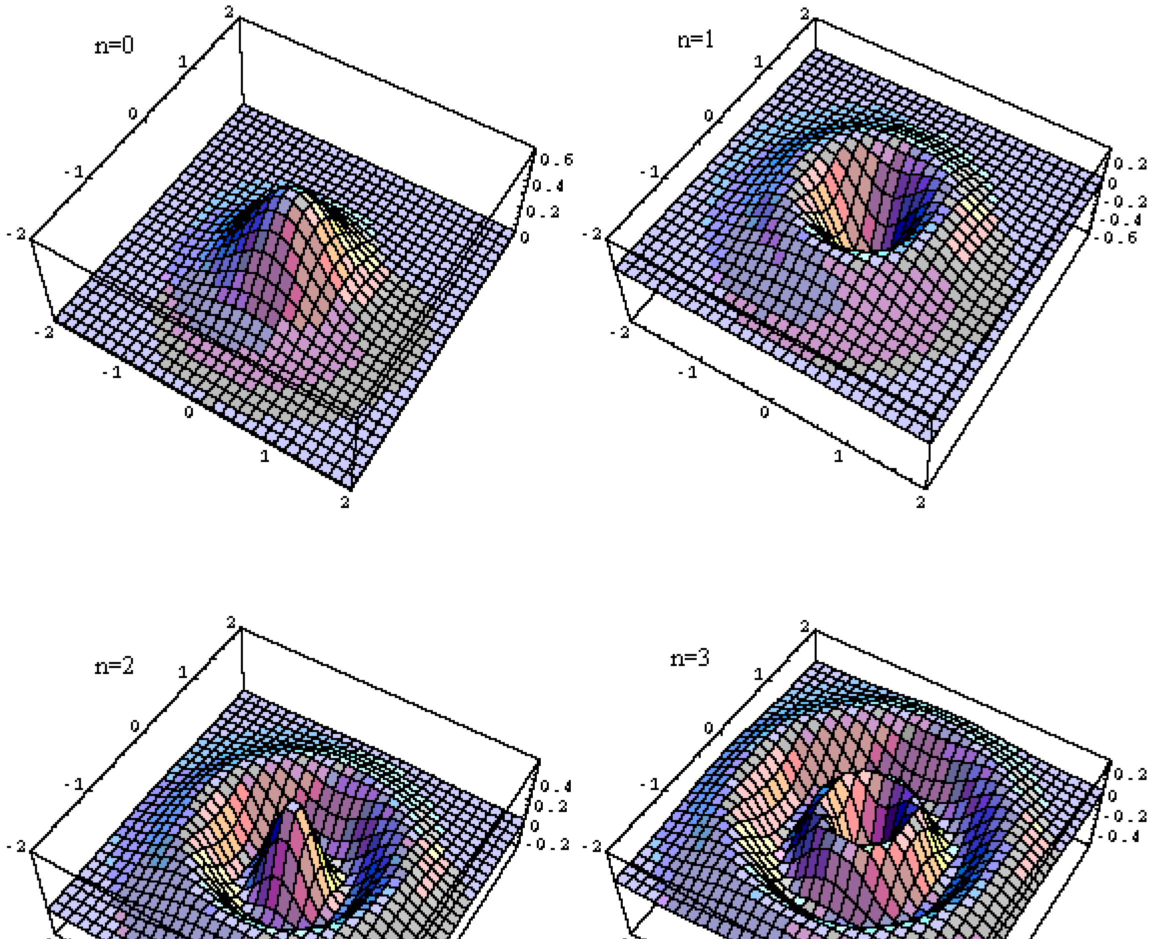


Figure 3. In computer-aided tomography (CAT) a beam of X-rays passes through a head, integrating the head's density function $D(x,y)$ along straight lines. If this is carried out for many different angles θ_n the density function can be reconstructed by computer processing of the projection data.


n=0
n=1
n=2
n=3

(b)

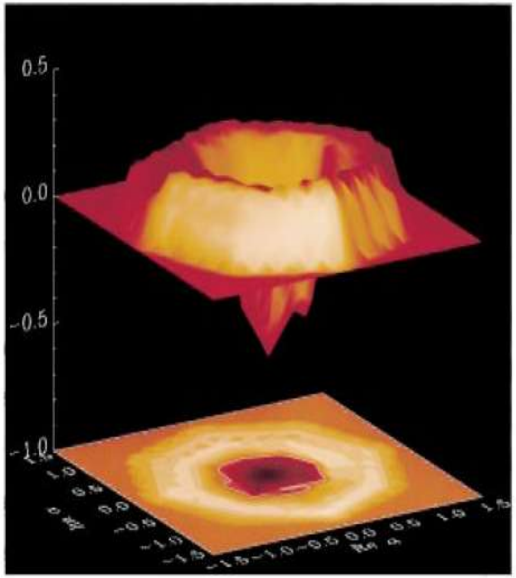


FIG. 1. (a) Reconstructed number-state density matrix amplitudes ρ_{nm} for an approximate $|n = 1\rangle$ number state. The coherent reconstruction displacement amplitude was $|\alpha| = 1.15(3)$. The number of relative phases $N = 4$ in Eq. (4), so $n_{\max} = 3$. (b) (color) Surface and contour plots of the Wigner function $W(\alpha)$ of the $|n = 1\rangle$ number state. The plotted points are the result of fitting a linear interpolation between the actual data points to a 0.1 by 0.1 grid. The octagonal shape is an artifact

signature. This view is further supported by the fact that farther off-diagonal elements seem to decrease faster than direct neighbors of the diagonal. The reconstructed Wigner function of a coherent state with amplitude $|\beta| \approx 1.5$ is shown in Fig. 3.

Next we created a coherent superposition of $|n = 0\rangle$ and $|n = 2\rangle$ number states. This state is ideally suited to demonstrate the sensitivity of the reconstruction to coherences. The only nonzero off-diagonal elements should be ρ_{02} and ρ_{20}, with a magnitude of $|\rho_{02}| = |\rho_{20}| = \sqrt{\rho_{00}\rho_{22}} \approx 0.5$ for a superposition with about equal probability of being measured in the $|n = 0\rangle$ or $|n = 2\rangle$ state. In the reconstruction shown in Fig. 4 the populations ρ_{00} and ρ_{22} are somewhat smaller, due to imperfections in the

From the Wigner function we can recover the wavefunction:

$$\psi(x)\psi(y)^* = \frac{1}{2\pi\hbar}\int W\left(\tfrac{x+y}{2}, p\right) e^{ip(x-y)/\hbar}\, dp$$

— choose a fixed y such that $\psi(y) \neq 0$.

linearly polarized, plane travelling wave can be represented at a fixed point in space by its oscillating electric-field amplitude,

$$E(t) = E_0[q \cos(\omega t + \theta) + p \sin(\omega t + \theta)]. \qquad (12)$$

Here E_0 is a known scaling field value that depends on the volume of the enclosure in which the light travels and Planck's constant $\hbar$. Here the variables called q and p have nothing to do with position or momentum; they are simply unitless numbers giving the strength of the electric field. The notation reminds us that q and p are complementary in the sense of equation (2). Also an optical phase shift θ is included in equation (12). Let us denote by q_0 and p_0 the values in the special case that $\theta = 0$. If the value of the phase shift is not zero, then q, p may be related to q_0 and p_0 by

$$q = q_0 \cos\theta + p_0 \sin\theta,$$

$$p = -q_0 \sin\theta + p_0 \cos\theta. \qquad (13)$$

This has a form similar to equation (11) and it also

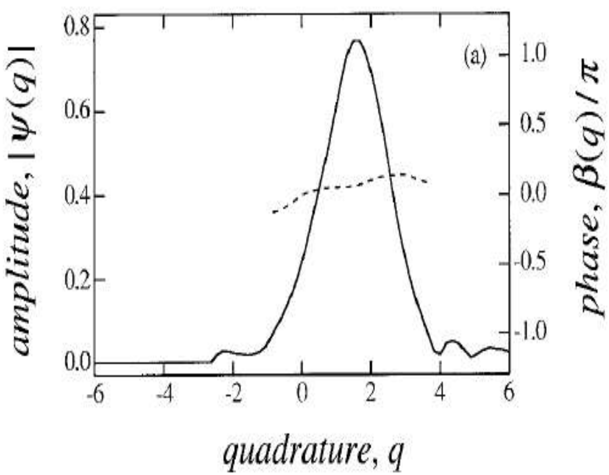


Figure 9. An experimentally reconstructed quantum wave function of a pulsed light field containing on average 1·2 photons, produced by strongly attenuating a laser field. The variable q is a unitless measure of electric-field strength, and $|\psi(q)|^2$ equals the probability for q to be found having a certain value. The wave function is expressed as $\psi(q) = |\psi(q)| \exp[i\beta(q)]$ where $|\psi(q)|$ is the amplitude (solid curve) and $\beta(q)$ is the phase structure (dashed curve). (From [20].)