Compact quantum systems: Internal geometry of relativistic systems

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A generalization is presented of the kinematical algebra so(5), shown previously to be relevant for the description of the internal dynamics (*Zitterbewegung*) of Dirac's electron. The algebra so($n+2$) is proposed for the case of a compact quantum system with n degrees of freedom. Associated wave equations follow from boosting these compact quantum systems. There exists a contraction to the kinematical algebra of a system with n degrees of freedom of the usual type, by which the commutation relations between n coordinate operators Q_i and corresponding momentum operators P_i, occurring within the so($n+2$) algebra, go over into the usual canonical commutation relations. The so($n+2$) algebra is contrasted with the sl(l,n) superalgebra introduced recently by Palev in a similar context: because so($n+2$) has spinor representations, its use allows the possibility of interpreting the half-integral spin in terms of the angular momentum of internal finite quantum systems. Connection is made with the ideas of Weyl on the possible use in quantum mechanics of ray representation of finite Abelian groups, and so also with other recent works on finite quantum systems. Possible directions of future research are indicated.

I. INTRODUCTION

Many years ago, Weyl[1] considered the unitary representation of the Lie group defined by Heisenberg's canonical commutation relations, and noted that it may also be considered as a ray representation of an infinite Abelian group. He speculated that unitary ray representations of finite Abelian groups might also prove important in quantum mechanics. Indeed, he gave the example of the unitary ray representation

$$g_1 \to i\sigma_1, \quad g_2 \to i\sigma_2, \quad g_3 \to i\sigma_3, \quad e \to I_2,$$

of the four-element Abelian group (Klein four-group), whose elements satisfy

$$\begin{aligned}&(g_1)^2 = (g_2)^2 = (g_3)^2 = e \text{ (identity)},\\ &g_2g_3 = g_3g_2 = g_1, \quad g_3g_1 = g_1g_3 = g_2,\\ &g_1g_2 = g_2g_1 = g_3,\end{aligned} \tag{1.1}$$

in connection with the description of the electron's spin. (Here the σ_i are Pauli matrices.)

Recent interest in "finite quantum systems" has approached the subject in three essentially different ways.

Santhanam and co-workers[2] have proceeded directly from Weyl's position, writing the unitary ray representatives of finite Abelian groups in exponential form in order to define finite-dimensional Hermitian analogs of Heisenberg's position and momentum variables, satisfying modified commutation relations. A related approach has been adopted by Gudder and Naroditsky,[3] and also by Stovicek and Tolar.[4]

Palev[5] has considered a simple dynamical system, the isotropic harmonic oscillator in n dimensions, and adopted a noncanonical quantization (in the spirit of Wigner's[6] well-known work, but along different lines) in order to arrive at noncanonical position and momentum variables with finite-dimensional representations.

Our own work[7] and continuing interest in this area has stemmed from the observation that Dirac's equation for the electron may be regarded as providing the covariant description of a finite quantum oscillator—the *Zitterbewegung*. Associated with this equation, in the rest frame of the electron's center of mass (or in any fixed frame with definite center of mass momentum), are internal coordinates Q_i and momenta $P_i (i = 1,2,3)$, which satisfy noncanonical commutation relations and have a finite (four-) dimensional Hermitian representation. The kinematical algebra generated by these three Q's and P's under commutation is isomorphic to the Lie algebra so(5).

The authors mentioned above, together with many others (see Jagannathan[8] and Saavedra and Utreras[9] for references), have speculated on the possible utility of novel kinematics in the description of the *internal* dynamics of real systems, and in particular, of some relativistic "particles." However, the so(5) algebra has the important distinguishing feature that it is *known* to be relevant to an important, real relativistic physical system, because of its association with Dirac's equation.[7]

Therefore, the structure of this particular kinematical algebra, its relation to the Heisenberg algebra and to Weyl's ideas, and its generalization to the case of n degrees of freedom (that is, n Q's and n P's) are of particular interest. This interest is heightened by the thought that the heavy leptons μ and τ may represent excited states of an internal electron dynamics. Furthermore, we show elsewhere that the cases $n=2$ and $n=4$, respectively, arise in the description of the internal dynamics of the neutrino,[10] and of the electron in a proper time formalism.[11]

II. THE KINEMATICAL ALGEBRA SO($n+2$)

In the description of the *Zitterbewegung* of the Dirac electron in the rest frame of its center of mass,[7] the three Hermitian operators Q_i appear as the coordinate of the charge relative to the center of mass. The three Hermitian operators P_i have been introduced as the corresponding rela-

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tive momentum variables. Together they generate the so(5) kinematical algebra, with commutation relations

$$[Q_i,Q_j] = (i\lambda^2/\hbar)S_{ij}, \tag{2.1a}$$

$$[P_i,P_j] = (4i\hbar/\lambda^2)S_{ij}, \tag{2.1b}$$

$$[Q_i,P_j] = i\hbar\delta_{ij}\, J, \tag{2.1c}$$

$$[Q_i,S_{jk}] = i\hbar(\delta_{ik}Q_j - \delta_{ij}Q_k), \tag{2.1d}$$

$$[P_i,S_{jk}] = i\hbar(\delta_{ik}P_j - \delta_{ij}P_k), \tag{2.1e}$$

$$[Q_i,J] = (i\lambda^2/\hbar)P_i, \tag{2.1f}$$

$$[P_i,J] = (4i\hbar/\lambda^2)Q_i, \tag{2.1g}$$

$$[J,S_{ij}] = 0, \tag{2.1h}$$

$$[S_{ij},S_{kl}] = i\hbar(\delta_{ik}S_{jl} + \delta_{jl}S_{ik} - \delta_{jk}S_{il} - \delta_{il}S_{jk}). \tag{2.1i}$$

Here λ is a constant with the dimension of length. As has been emphasized before,[7] the appearance of at least one such constant is inevitable in any finite quantum system incorporating Hermitian coordinate variables, whose eigenvalues are necessarily discrete, with dimensions of length. In the application of the so(5) algebra to the internal dynamics of the electron, λ equals the Compton wavelength of that particle. Furthermore, in that application the operators of the algebra (2.1) can be expressed in terms of the more familiar Dirac matrices as

$$Q_i = \tfrac{1}{2}i\lambda\alpha_i\,\beta, \tag{2.2a}$$

$$P_i = (\hbar/\lambda)\alpha_i, \tag{2.2b}$$

$$J = -\beta, \tag{2.2c}$$

while S_{ij} is the usual spin tensor

$$S_{ij} = -\tfrac{1}{4}i\hbar[\alpha_i,\alpha_j] = \epsilon_{ijk}S_k. \tag{2.2d}$$

The relevant representation of so(5) is then the four-dimensional spinor representation, in which $J(=-\beta)$ is a traceless operator with unit square.

There is an obvious generalization of the algebra (2.1) to the case of n degrees of freedom: simply allow the indices there to run over $1,2,...,n$ instead of 1,2,3. Then the Lie algebra so$(n+2)$ is obtained. If one defines $J_{AB}(=-J_{BA}$, $A,B = 1,2,...,n+2)$ by setting $J_{ij} = S_{ij}/\hbar$, $J_{i,n+1} = \lambda^{-1}Q_i$, $J_{i,n+2} = (\lambda/2\hbar)P_i$, and $J_{n+1,n+2} = \frac{1}{2}J$, then the J_{AB} satisfy the so$(n+2)$ commutation relations in standard form

$$[J_{AB},J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}). \tag{2.3}$$

The fundamental spinor representations of so$(n+2)$, of dimension 2^p, are of particular interest. [Here $p = \frac{1}{2}(n+1)$ if n is odd, and $p = \frac{1}{2}n$ if n is even. In the latter case there are two inequivalent representations.] The relations of such representations to Clifford algebras, and associated anticommutation relations, are well known. Only in these representations does the operator J, which is traceless in every representation, have unit square, so that its eigenvalues are ± 1. Inspection of (2.1c) suggests that one is then, in an intuitive sense, as close as possible to the canonical commutation relations

$$[q_i,p_j] = i\hbar\delta_{ij}I, \tag{2.4}$$

where I is the unit operator. (Note that the commutator of any Q_i and P_j represented by finite matrices must be traceless.)

Various dynamics are possible within the framework of the so$(n+2)$ algebra, corresponding to various choices of Hamiltonian operator H in the enveloping algebra of the particular representation at hand. In the case $n=3$, when the fundamental (Dirac) spinor representation is chosen, the only true so(3) scalars available (as distinct from pseudoscalars) are $J(=-\beta)$ and I (identity). With H of the form $cI + d\beta$, where c and d are numbers with dimensions of energy, the commutation relations (2.1f) and (2.1g), together with Heisenberg's equation of motion

$$i\hbar\dot{A} = [A,H], \quad \dot{A} = \frac{dA}{dt}, \tag{2.5}$$

imply

$$\dot{Q}_i = d(\lambda^2/\hbar^2)P_i, \quad \dot{P}_i = -(4d/\lambda^2)Q_i, \tag{2.6}$$

so that

$$\ddot{Q}_i = -(4d^2/\hbar^2)Q_i, \quad \ddot{P}_i = -(4d^2/\hbar^2)P_i. \tag{2.7}$$

Thus harmonic oscillator dynamics is singled out in this case.[7] This would not be true for other representations of so(5), nor for larger values of n, even in the fundamental spinor representations.

Nevertheless, because the constants h and λ are available, dimensionless creation and annihilation operators can *always* be defined, whatever the representation and whatever the dynamics, as

$$\begin{aligned} A_i &= Q_i/\lambda + i(\lambda/2\hbar)P_i, \\ A_i^\dagger &= Q_i/\lambda - i(\lambda/2\hbar)P_i, \quad i = 1,2,...,n. \end{aligned} \tag{2.8}$$

The $A_i^\dagger$ is Hermitian conjugate to A_i, and relations (2.1) become

$$\begin{aligned} &[A_i, A_j] = 0 = [A_i^\dagger, A_j^\dagger], \\ &[A_i, A_j^\dagger] = \delta_{ij}J + (2i/\hbar)S_{ij}, \\ &[A_i, J] = -2A_i, \quad [A_i^\dagger, J] = +2A_i^\dagger, \end{aligned} \tag{2.9}$$

together with (2.1h) and (2.1i) and relations like (2.1d) and (2.1e), which express the n-vector nature of A_i and $A_i^\dagger$.

The relations (2.1) are also equivalent to

$$\begin{aligned} &[[A_i, A_j^\dagger], A_k] = 2(\delta_{ij}A_k + \delta_{jk}A_i - \delta_{ik}A_j), \\ &[[A_i, A_j^\dagger], A_k^\dagger] = 2(-\delta_{ij}A_k^\dagger + \delta_{jk}A_i^\dagger - \delta_{ik}A_j^\dagger), \\ &[[A_i, A_j^\dagger], [A_k, A_l^\dagger]] \\ &\quad = 2(\delta_{ik}[A_l, A_j^\dagger] - \delta_{lj}[A_i, A_k^\dagger] \\ &\quad + \delta_{jk}[A_i, A_l^\dagger] - \delta_{il}[A_k, A_j^\dagger]) \\ &[A_i, A_j] = 0 = [A_i^\dagger, A_j^\dagger], \end{aligned} \tag{2.10}$$

in which form they show most clearly how these operators differ from the ones introduced for a finite quantum oscillator by Palev.[5] His operators satisfy

$$\begin{aligned} &[\{A_i^\dagger, A_j\}, A_k] = -\delta_{ik}A_j + \delta_{ij}A_k, \\ &[\{A_l^\dagger, A_j\}, A_k^\dagger] = \delta_{jk}A_i^\dagger - \delta_{ij}A_k^\dagger, \\ &[\{A_i^\dagger, A_j\}, \{A_k^\dagger, A_l\}] = \delta_{jk}\{A_i^\dagger, A_l\} - \delta_{il}\}\{A_k^\dagger, A_j\}, \\ &\{A_i, A_j\} = 0 = \{A_i^\dagger, A_j^\dagger\}, \end{aligned} \tag{2.11}$$

and define the Lie superalgebra sl(l,n). Like so($n+2$), this has infinitely many inequivalent irreducible Hermitian representations.

Palev considered his algebra as a dynamical algebra associated with a particular Hamiltonian

$$H = [\hbar\omega/(n-1)]\{A_i{}^\dagger, A_i\}, \tag{2.12}$$

for an isotropic oscillator. [Here the constant ω is introduced, with dimensions of (time)$^{-1}$, but Palev also needs to introduce a constant with dimensions of a length in order to define coordinate and momentum operators.] In contrast, we view the so($n+2$) algebra as kinematical. It always admits as a *particular* dynamics, that associated with the Hamiltonian

$$H = (\hbar\omega/2n)[A_i{}^\dagger, A_i], \tag{2.13}$$

which leads to the harmonic oscillator equations

$$\dot{A}_i = -i\omega A_i, \quad \dot{A}_i{}^\dagger = +i\omega A_i{}^\dagger, \tag{2.14}$$

or equivalently, to Eqs. (2.7) with

$$d = \tfrac{1}{2}\hbar\omega. \tag{2.15}$$

As already remarked, this is the only dynamics permitted in the case of the fundamental spinor representation of so(5) ($n = 3$), when it is directly relevant to the description of the *Zitterbewegung* of the electron as a finite quantum oscillator.[7] No doubt Palev's algebra (without reference to ω) could also be viewed more widely as a kinematical algebra admitting a variety of representations, and a variety of dynamics in most representation.

Another important distinction between the so($n+2$) and sl(l,n) algebras relates to the representations of the so(n) subalgebra that can appear. This subalgebra is associated in both cases with the "angular momentum" of the finite quantum system. Since spinor representation of so($n+2$) are allowed (as for the electron), then spinor representations of the so(n) subalgebra can be accomodated. However, the so(n) subalgebra of sl(l,n) appears in the chain

$$\mathrm{so}(n) < \mathrm{sl}(n) < \mathrm{sl}(l,n), \tag{2.16}$$

and only tensor representations of so(n) appear in the representations of sl(n). Thus Palev's algebra can only describe finite quantum systems with integral angular momentum or spin.

We comment at the end about the noncompact versions of the so($n+2$) algebras.

III. RELATIONSHIP TO THE HEISENBERG ALGEBRA AND TO WEYL'S IDEA

The so($n+2$) algebra, which is of dimension $\frac{1}{2}(n+1)(n+2)$, is generated by the n Q's and n P's under commutation, as Eqs. (2.1) show. In contrast, n canonical Q's and P's generate the Heisenberg algebra, which is of the smaller dimension ($2n+1$):

$$[q_i q_j] = 0 = [p_i, p_i], \quad [q_i p_i] = i\hbar\,\delta_{ij} I, \qquad [I, q_i] = 0 = [I, p_i]. \tag{3.1}$$

These may be compared with Eqs. (2.1a)–(2.1c), (2.1e), and (2.1f). However, it is more appropriate to compare the so($n+2$) algebra with the kinematical Lie algebra k_n, also of dimension $\frac{1}{2}(n+1)(n+2)$, obtained by extending the Heisenberg algebra by the algebra so(n) of rotations; introduce the $\frac{1}{2}n(n-1)$ so(n) (angular momentum) operators $l_{ij}\,(=-l_{ji}, i,j = 1, 2,\ldots,n)$ satisfying

$$\begin{aligned} &[q_i, l_{jk}] = i\hbar(\delta_{ik} q_j - \delta_{ij} q_k),\\ &[p_i, l_{jk}] = i\hbar(\delta_{ik} p_j - \delta_{ij} p_k), \quad [I, l_{ij}] = 0,\\ &[l_{ij}, l_{kl}] = ih\,(\delta_{ik} l_{jl} + \delta_{jl} l_{ik} - \delta_{jk} l_{il} - \delta_{il} l_{jk}), \end{aligned} \tag{3.2}$$

which may be compared with Eqs. (2.1d), (2.1e), (2.1h), and (2.1i). Any representation of the Heisenberg algebra can be extended to a representation of k_n by setting

$$l_{jk} = q_j p_k - q_k p_j. \tag{3.3}$$

However, there are also representations of k_n in which the relation (3.3) does not hold. We may always add one or more "spin terms" to the right-hand side of Eq. (3.3), thus ensuring in particular that spinor representations of so(n) can occur.

It is noteworthy that, although there is (up to equivalence) only one unitary representation of the (Weyl) group associated with the Heisenberg Lie algebra, by von Neumann's theorem, there are evidently infinitely many inequivalent unitary representations (with various spin content) of the group K_n whose Lie algebra is k_n. Corresponding to this in our case is the fact that there are infinitely many inequivalent unitary representations of the group SO($n+2$).

There is a contraction[12] from the algebra so($n+2$) to k_n; this emphasizes the naturalness of the choice of so($n+2$) as an appropriate kinematical algebra for finite quantum systems. To see this without going into details, define

$$\tilde{q}_i = \epsilon_1 Q_i, \quad \tilde{p}_i = \epsilon_2 P_i, \quad \tilde{I} = \epsilon_1\epsilon_2 J, \quad \tilde{l}_{ij} = S_{ij}, \tag{3.4}$$

with Q_i, p_i, etc., as in (2.1) and ϵ_1, ϵ_2 real parameters. Then

$$\begin{aligned} &[\tilde{q}_i, \tilde{q}_j] = i(\lambda^2/\hbar)(\epsilon_1)^2 l_{ij},\\ &[\tilde{p}_i \tilde{p}_j] = (4i\hbar/\lambda^2)(\epsilon_2)^2 l_{ij},\\ &[\tilde{q}_i, \tilde{p}_j] = i\hbar\delta_{ij}\tilde{I},\\ &[\tilde{q}_i, \tilde{I}] = -i(\lambda^2/\hbar)(\epsilon_1)^2 \tilde{p}_i,\\ &[\tilde{p}_i, \tilde{I}] = (4i\hbar/\lambda^2)(\epsilon_2)^2 \tilde{q}_i, \end{aligned} \tag{3.5}$$

while the remaining relations are as in Eqs. (3.2), with q_i replacing q_i, etc. When ϵ_1 and ϵ_2 are set to zero, Eqs. (3.5) reduce to Eqs. (3.1). If ϵ_1 is set to zero but not ϵ_2 (or vice versa), the Lie algebra obtained can be seen to be that of the Euclidean group $E(n+1)$. (These cases correspond physically to an oscillator or free particle.) This indicates that the contraction from so($n+2$) to k_n can proceed in two stages, via $e(n+1)$ (and that there are two distinct routes along which this may be accomplished).

There is also a close relationship between the fundamental spinor representations of the so($n+2$) algebra, and unitary ray representations of finite Abelian groups, so that contact can be made with Weyl's idea,[1] and also the work of Santhanam,[2] mentioned in the Introduction. Consider, for example, the case $n = 1$ (one Q and one P) and the fundamental spinor representation of so(3), which is two dimensional. We may take in this case

$$Q = \tfrac{1}{2}\lambda\sigma_1, \quad P = (\hbar/\lambda)\sigma_2, \tag{3.6}$$

where σ_1 and σ_2 are Pauli matrices. Then Eqs. (2.1) show

$$J = \sigma_3, \tag{3.7}$$

while there are no so(n) operators in this case. Define the unitary operators

$$A(\theta) = \exp(i(\theta/\lambda)Q), \quad B(\phi) = \exp(i(\phi\lambda P/2\hbar)). \quad (3.8)$$

Then

$$A(\pi) = i\sigma_1, \quad B(\pi) = i\sigma_2, \quad (3.9)$$

and it can be seen that $A(\pi)$ and $B(\pi)$ generate under multiplication the unitary ray representation of the four-element Abelian group defined by Eqs. (1.1). In contrast, the set of all unitary operators $A(\theta)$, $B(\phi)$, with $\theta,\phi \in [0,2\pi]$, generate under multiplication a two-valued representation of SO(3) [that is, a true representation of SU(2)].

Note that if we started with the unitary ray representation of the Abelian group, and hence with $A(\pi)$ and $B(\pi)$, we could define

$$Q = -(i\lambda/\pi)\log A(\pi), \quad P = -(2i\hbar/\lambda)\log B(\pi), \quad (3.10)$$

and recover the so(3) algebra generated under commutation by Q and P. On the other hand, if we started with a unitary representation of su(2), we would more naturally identify Q and P by setting

$$Q = -i\lambda \left.\frac{dA(\theta)}{d\theta}\right|_{\theta=0}, \quad P = -\frac{i\lambda}{2\hbar} \left.\frac{dB(\phi)}{d\phi}\right|_{\phi=0}. \quad (3.11)$$

IV. CONCLUDING REMARKS

Of various approaches to the description of a finite quantum system with n degrees of freedom, the one using the so($n+2$) kinematical algebra is distinguished primarily by the fact that it is known to be relevant to real relativistic systems.[7,10,11] Furthermore, it has been shown that there is a well-defined relationship between the so($n+2$) algebra and the kinematical algebra k_n of a system with n degrees of freedom of the usual (noncompact) type. This relationship is defined by a group contraction.

Of course, we do not claim that so($n+2$) is the only algebra which could have such a relationship with k_n. However, the existence of this relationship suggests the possibility of studying a class of finite quantum systems which are well-defined analogs of infinite quantum systems, and also the connection between the two, through the contraction process. One could start with the finite quantum oscillator, as in Eqs. (2.7), for example, but it would be interesting also to construct finite analogs of other well-known dynamical systems, such as the Kepler system, and to investigate their symmetry and dynamical algebras.

Another important distinguishing feature of the so($n+2$) algebra which has been emphasized above is the existence of spinor representations. This makes possible the "explanation" of the half-integral spin of "elementary" particles as the angular momentum of internal finite quantum systems. Such an idea dates back to Schrödinger's work on Dirac's electron,[13] and has been further brought out in our own recent efforts.[7]

Finite systems can be accommodated naturally in the vector space setting of quantum mechanics—we merely need to consider finite-dimensional subspaces of Hilbert space. On the other hand, one might suppose that they have no classical counterparts. That this is not necessarily the case is shown, for example, by the recent construction of a classical analog of Dirac's spinning electron.[14] (In this connection, we mention also the earlier work by Grossmann and Peres.[15])

There is clearly more to be done towards understanding the relationship of finite quantum systems to the more familiar dynamical systems of classical and quantum mechanics. The use of the so($n+2$) kinematical algebra defines a class of finite systems for which some possible directions of future research seem reasonably well defined.

Once the commutation relations of the internal dynamical variables have been recognized, we can also take the infinite-dimensional representations of the internal algebra so($n+2$). These then represent many-body systems with n degrees of freedom in the center of mass frame. Relativistic theories of composite atoms or hadrons,[16] or relativistic oscillator and rotator,[17] belong to this category. The boosting of such a system (i.e., induced representations of the Poincaré group) gives relativistic finite-component wave equations in the case of finite-dimensional representations, and infinite-component wave equations in the case of composite systems.

In the infinite-dimensional case one can use perhaps more appropriately the unitary representations of the noncompact form of the algebras so(p,q). The exact form of the noncompact form depends on the physical interpretation of the generators as Hermitian operators. For example, the so(3,2) form of so(5) has been used extensively.[11,16–18]

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